609 Homework 3

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March 21, 2013

(5.9) The following algorithms yields the desired result. If it does not, then interchange the roles of S and T.

Algorithm 1 Algorithm for finding a matching M'', given matching M and M' in a bipartite graph

Input: $S \subseteq A, T \subseteq B, M, M'$
Output: M'' (a matching for $S \cup T$)

$M'' = \emptyset; S' = \emptyset$
for each $s \in S$ **do**
  Let $\{s, t\} \in M$ for some $t \in B$ be the matching of s in M.
  if $t \in T$ **then**
    $M'' = M'' \cup \{\{s, t\}\}$
  else
    $S' = S' \cup \{s\}$
  end if
end for
for each $t \in T$ **do**
  if $\nexists\{s, t\} \in M''$ (i.e., t is not matched by M'') **then**
    Let $\{s, t\} \in M'$ for some $s \in A$ be the matching of t in M'.
    $M'' = M'' \cup \{\{s, t\}\}$
  end if
end for
for each $s' \in S'$ **do**
  if $\nexists\{s', t\} \in M''$ (i.e., $s' \in S$ is not matched by M'') **then**
    Let $\{s', t\} \in M'$ for some $t \in B$ be the matching of s' in M.
    $M'' = M'' \cup \{\{s', t\}\}$
  end if
end for

The above algorithm partitions the vertices of S into two disjoint sets according to M: vertex connected to members of T and the rest. Every vertex connected to a member of T is included in M''. The rest is added to the set S' to consider later. Next, every vertex of T that has not been matched before is matched according to M', so every vertex of T is included in M''. Finally, the edges corresponding to vertices in S' according to M are included if and only if they have not been included before. Hence, every vertex in S is included. Note that it is possible for vertices outside of S and T to be matched, but we know that at least all vertices in S and T are matched. Moreover, the algorithm does not include non-disjoint edges since it includes vertices if and only if one of the vertices has not been included before. The result is stored in M''.

(5.1) Suppose that $S_1, S_2, \cdots, S_m$ does not have a System of Distinct Representatives. Then, by Hall's theorem, the union $Y = S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_k}$ of some k $(1 \leq k \leq m)$ sets contains strictly less than k elements. For $x \in Y$, let d_x be the number of sets containing x. Using the double counting argument in (1.10):

$$k \cdot r \leq \sum_{j=1}^{k} |S_{i_j}| = \sum_{x \in Y} d_x \leq r|Y| < k \cdot r$$

From which we get that $k \cdot r < k \cdot r$, a clear contradiction. Hence, $S_1, S_2, \cdots, S_m$ have a SDR. □

(5.4) Since $S_1, S_2, \cdots, S_m$ satisfy Hall's condition, they have a SDR. By hypothesis, we have that $|S_1 \cup \cdots \cup S_k| = k$.
Claim: the first k sets are singletons, each containing a distinct element, i.e., $S_i = \{x_i\}$ for all $i = 1, \cdots, k$ where $x_i \neq x_j$ for all distinct $1 \leq i, j \leq k$.
Proof: By definition of SDR is clear that each set must contain at least one element. Now, suppose that not all S_i, $i = 1, \cdots, k$ are singletons. Then there exists one set that contains more than one element. But then $|S_1 \cup \cdots \cup S_k| > k$, a contradiction. □ (of claim)
Finally, since $S_1, S_2, \cdots, S_m$ have a SDR, it follows that each one of $S_{k+1}, \cdots, S_m$ have at least one element distinct from all other sets. Hence, none of these can lie entirely in the above union. □

(5.6) Let $G = (A, B, E)$ be a bipartite graph. Let a be the minimum degree of a vertex in A and b the maximum degree of a vertex in B. Suppose that $a \geq b$. Also, suppose (for a contradiction) that there does not exits a matching of A into B. Define for each $x \in A$ the set $S_x = \{y \in B : \{x, y\} \in E\}$. By theorem 5.6, it follows that there exists a subset of k vertices from A with less than k neighbors, i.e., a set $Y = S_{x_{i_1}} \cup \cdots \cup S_{x_{i_k}}$ for some k $(1 \leq k \leq |A|)$ such that $|Y| < k$. For $x \in Y$, let d_x be the number of sets containing x. But then, by double counting argument (1.10):

$$k \cdot a \leq \sum_{j=1}^{k} |S_{x_{i_j}}| = \sum_{x \in Y} d_x \leq |Y| b < k \cdot b$$

From which we conclude that $a < b$, in contradiction with our initial hypothesis. Hence, every subset ok k vertices from A has at least k neighbors which means that G has a matching of A into B. □

(5.10) For $I \subseteq A$, let $S(I) \subseteq B$ be the set of neighbors of I in G. On the one hand, let us show that the set $A' = (A \setminus I) \cup S(I)$ intersects all the members of $\mathcal{F}$. Suppose not. Then there exists $I \subseteq A$ and $F \in \mathcal{F}$ such that $A' \cap F = \emptyset$. But then:

$\emptyset$	$=$	$A' \cap F$	
	$=$	$[(A \setminus I) \cup S(I)] \cap F$	By definition of A'
	$=$	$[(A \setminus I) \cap F] \cup [S(I) \cap F]$	By distributivity of sets
	$\iff$	$(A \setminus I) \cap F = \emptyset$ and $S(I) \cap F = \emptyset$	(1) Since union of empty sets yield the empty set

Since by hypothesis A intersects all the members of $\mathcal{F}$ and F in particular, we have that

$$(A \setminus I) \cap F = \emptyset \Rightarrow I \cap F \neq \emptyset$$

Moreover, by hypothesis B intersects every set in $\mathcal{F}$, so in particular it intersects F. Therefore, since $|F| \geq 2$, we can conclude that there exists a connection between a member of $a \in I$ and $b \in B$, which means that $S(I) \cap F \neq \emptyset$, a contradiction with (1). It follows that the set A' intersects all the members of $\mathcal{F}$.

On the other hand, by Theorem 5.6 (Hall's), to show that G has a matching it suffices to show that for every $k = 1, 2, ..., |A|$, every subset of k vertices from A has at least k neighbors. Once again, suppose that this is not the case. Then, there exists $\emptyset \neq I \subseteq A$ such that $|S(I)| < |I|$. Take this set I and consider A' as defined before but for this particular I, i.e., $A' = (A \setminus I) \cup S(I)$. We showed that A' must intersect every member of $\mathcal{F}$. But then, since $A \cap B = \emptyset$, we have that $|A'| = |A| - |I| + |S(I)| < |A|$, contradicting the assumption that no set of fewer than $|A|$ elements intersects every member of $\mathcal{F}$. Hence, Hall's theorem hold, which means that G has a matching of A into B, but since $|A| = |B|$, this is a perfect matching. □

(6.2) Suppose that $\mathcal{F}$ has a sunflower with k petals. Let $S_1, \cdots, S_k$ be the petals. The core Y is such that $0 \leq |Y| \leq s - 1$, but this is not possible. Suppose $|Y| = 0$. By definition of sunflower $S_1 \cap S_2 \cap \cdots \cap S_k = \emptyset$, but this implies that $|V_i| = k$, for all $1 \leq i \leq s$, a contradiction. Likewise, suppose $|Y| = 1$. By definition of sunflower $S_1 \setminus Y \cap S_2 \setminus Y \cap \cdots \cap S_k \setminus Y = \emptyset$. But then, assuming that this 1 element came from V_i we must have that $|V_j| = k$ for any $j \neq i$, a contradiction.
In general, for any candidate core Y such that $0 \leq |Y| \leq s-1$, let us define $I := \{j : y \in V_j \text{ for } y \in Y\}$. Note that since the sets V_i are pairwise disjoints we have that $|I| \leq s - 1$. But then, it is the case that $|V_l| = k$ for any $l \notin I$, a contradiction since there is at least one set V such that $|V| = k$.
Note that if $|Y| = s$, then either Y contains distinct elements from each of the V_i sets, in which case $Y \in \mathcal{F}$, and it cannot possibly be a valid core; or Y does not contain elements from all distinct sets and the previous analysis applies, i.e., there is at least one V_i such that $|V_i| = k$, a contradiction.
Therefore, the family $\mathcal{F}$ does not contain a sunflower with k petals. □

(6.3)

<u>For Lemma 6.3</u>: It suffices to show that for the family $\mathcal{F}$ as defined in 6.2, the common part of every k members of $\mathcal{F}$ has at least s elements. Recall that $\mathcal{F}$ is an s-uniform family such that $|\mathcal{F}| = (k-1)^s$.
Suppose to the contrary that there exists k sets in $\mathcal{F}$ such that the common part of these sets have less than s elements, i.e., $S_1, \cdots, S_k$ such that $|Y| = |\bigcup_{i \neq j} S_i \cap S_j| < s$. By definition, $S_1 \setminus Y \cap \cdots \cap S_k \setminus Y = \emptyset$
Since $|Y| < s \iff 0 \leq |Y| \leq s - 1$. Now apply the same reasoning as in 6.2 to conclude that this is not possible (e.g., if $|Y| = 0$ then $|V_i| = k$, for all $1 \leq i \leq s$, etc). Therefore, the common part of every k members of $\mathcal{F}$ has at least s elements. This is the optimal bound since if we add one more set to $\mathcal{F}$, we get that $|\mathcal{F}| > (k-1)^s$, and then apply Lemma 6.3 to obtain the desired result. □

For Lemma 6.4: It suffices to show that the family $\mathcal{F}$ as defined in 6.2 has no flower with k petals. As before, Recall that $\mathcal{F}$ is an s-uniform family such that $|\mathcal{F}| = (k-1)^s$.

Suppose to the contrary that $\mathcal{F}$ has a flower with k petals and a core Y. Just like before, we know that $0 \leq |Y| \leq s-1$. This is because we know that if $|Y| = s$, then Y contains a distinct element from each V_i or it repeats element from some V_i. On the one hand, If it repeats an element, then $Y \nsubseteq F$ for any $F \in \mathcal{F}$. Hence $\emptyset \subseteq \mathcal{F}_Y$ and then $\tau(\mathcal{F}_Y) = 0 < k$. On the other, if Y contains a distinct element from each V_i, then $Y \in \mathcal{F}$ and then $\emptyset \subset \mathcal{F}_Y$ and then $\tau(\mathcal{F}_Y) = 0 < k$. Either case is not possible, so $0 \leq |Y| \leq s-1$. But any of these are also not possible. Let us explore some of these cases:

Suppose $|Y| = 0$. Then $\mathcal{F}_Y = \mathcal{F}$, which means that $\tau(\mathcal{F}_y) = \tau(\mathcal{F}) = k-1 < k$, just take one of the V_i as your blocking set.

Suppose that $|Y| = 1$. Then this one element must come from one V_i. But then, $\tau(\mathcal{F}_Y) = k-1 < k$, just take one of the V_j as your blocking set subject to $i \neq j$ (take another set besides the one where the element of Y came from).

In general, let $|Y| = n$, $0 \leq n \leq s-1$. Define $I := \{j : y \in V_j \text{ for } y \in Y\}$. Then, take as your blocking set any set V_l such that $l \notin I$. But then, $\tau(\mathcal{F}_Y) = |V_l| = k-1 < k$, a contradiction.

Therefore, $\mathcal{F}$ has no sunflower with k petals. This is the optimal bound since if we add one more set to $\mathcal{F}$, we get that $|\mathcal{F}| > (k-1)^s$, and then apply Lemma 6.4 to obtain the desired result. □

(6.6) Suppose that $\mathcal{F}$ has a sunflower with k petals. Then, since $|S_i| = s$ for all $i = 1, ..., k$ and $S_1 \setminus Y \cap \cdots \cap S_k \setminus Y = \emptyset$ the number of elements used in this sunflower is $k(s - |Y|) + |Y|$, where Y is the sunflower's core and $0 \leq |Y| \leq s-1$. In words, add to the number of elements in all petals the number of elements in the core. Clearly, the number of elements in the sunflower cannot exceed the total number of elements used to build the sets in the family $\mathcal{F}$, i.e., $n \geq k(s-|Y|)+|Y| = ks-|Y|(k-1)$. Also, by hypothesis, $n-k+1 < s \Rightarrow n < s+k-1$. But then,

$$s+k-1 > n \geq ks - |Y|(k-1) \Rightarrow s+k-1 > ks - |Y|(k-1)$$

Subtract s from both sides of the last inequality to get $k-1 > s(k-1) - |Y|(k-1) = (k-1)(s-|Y|)$. So, $k-1 > (k-1)(s-|Y|)$, but since $0 \leq |Y| \leq s-1$, we get a contradiction, showing that $\mathcal{F}$ has no sunflower with k petals. □