609 Homework 4

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(8.1) Let $\mathcal{F}$ be an antichain consisting of sets of size at most $k \leq \frac{n}{2}$. Note that n is a fix number. The crucial point here is that the binomial coefficient is an increasing function over the interval $[0, \frac{n}{2}]$. By hypothesis, for any $A \in \mathcal{F}$ we have that $|A| \leq k \leq \frac{n}{2}$. Hence, for any given $A \in \mathcal{F}$:

$$\binom{n}{k} \geq \binom{n}{|A|}$$

$$\binom{n}{k}^{-1} \leq \binom{n}{|A|}^{-1} \quad \text{Inverting both sides}$$

$$\sum_{A\in\mathcal{F}} \binom{n}{k}^{-1} \leq \sum_{A\in\mathcal{F}} \binom{n}{|A|}^{-1} \quad \text{Summing over all elements of } \mathcal{F} \text{ in both sides}$$

$$|\mathcal{F}|\binom{n}{k}^{-1} \leq \sum_{A\in\mathcal{F}} \binom{n}{|A|}^{-1} \quad \text{Rewritting the left-hand side sum}$$

$$|\mathcal{F}|\binom{n}{k}^{-1} \leq \sum_{A\in\mathcal{F}} \binom{n}{|A|}^{-1} \leq 1 \quad \text{LYM inequality}$$

$$|\mathcal{F}|\binom{n}{k}^{-1} \leq 1 \quad \text{Now, multiply by } \binom{n}{k} \text{ both sides}$$

$$|\mathcal{F}| \leq \binom{n}{k} \quad \text{Obtaining the result.} \quad \Box$$

(8.4) Let $0 < p < 1$ be a real number and $C \subset D$ be any two fixed subsets of $\{1, 2, ..., n\}$. Then, summing over all sets $C \subseteq A \subseteq D$ we obtain:

$$\begin{aligned}
\sum_{C\subseteq A\subseteq D} p^{|A|}(1-p)^{n-|A|} &= \sum_{k=0}^{|D|-|C|} \binom{|D|-|C|}{k} p^{|C|+k}(1-p)^{n-(|C|+k)} && \text{Making the change } |A| = |C| + k \\
&= p^{|C|}(1-p)^{n-|C|} \sum_{k=0}^{|D|-|C|} \binom{|D|-|C|}{k} (\tfrac{p}{1-p})^k \cdot 1^{|D|-|C|-k} && \text{Rearranging terms} \\
&= p^{|C|}(1-p)^{n-|C|}(\tfrac{p}{1-p} + 1)^{|D|-|C|} && \text{By Binomial Theorem} \\
&= p^{|C|}(1-p)^{n-|C|}(\tfrac{1}{1-p})^{|D|-|C|} && \text{Summing fraction} \\
&= p^{|C|}(1-p)^{n-|C|}(1-p)^{|C|-|D|} && \text{Rearranging power} \\
&= p^{|C|}(1-p)^{n-|D|} && \text{Summing exponents} \quad \Box
\end{aligned}$$

(8.5) Let $\mathcal{F}$ be a k-uniform family, and suppose that it is intersection free.

Fix a $B_0 \in \mathcal{F}$ and form the family $\mathcal{C} = \{A \cap B_0 : A \in \mathcal{F}, A \neq B_0\}$. Claim: $\mathcal{C}$ is an antichain over B_0. Proof: suppose not: then there exists $C_1 \in \mathcal{C}$ and $C_2 \in \mathcal{C}$ such that $C_1 \subseteq C_2$. By definition $C_1 = A_i \cap B_0 \subseteq A_j \cap B_0 = C_2$, for some $A_i \in \mathcal{F}$ and $A_j \in \mathcal{F}$. But if $A_i \cap B_0 \subseteq A_j \cap B_0$ then $A_i \cap B_0 \subseteq A_j$ contradicting the hypothesis that $\mathcal{F}$ is intersection free. Hence, $\mathcal{C}$ is an antichain over B_0. $\Box$(*of claim*)

Since $\mathcal{C}$ is an antichain over B_0 where $|B_0| = k$, by Sperner's Theorem we know that $|\mathcal{C}| \leq \binom{k}{\lfloor k/2 \rfloor}$.

Also, since $\mathcal{F}$ is an intersection free family, then $|\mathcal{C}| = |\mathcal{F}| - 1$, i.e., the family $\mathcal{C}$ contains as many elements (subsets) as $\mathcal{F}$ except for B_0. This is because $A_i \cap B_0 \not\subset A_j$ which means that for any C_1, C_2 in $\mathcal{C}$, $C_1 = A_i \cap B_0 \not\subset A_j \cap B_0 = C_2$, so these are all distinct sets inside $\mathcal{C}$. Therefore:

$$|\mathcal{C}| = |\mathcal{F}| - 1 \leq \binom{k}{\lfloor k/2 \rfloor} \Longrightarrow |\mathcal{F}| \leq 1 + \binom{k}{\lfloor k/2 \rfloor}$$

□

(13.1) Let x, y be orthogonal vectors in a vector space. Then:

$$\begin{aligned}
||x+y||^2 &= < x+y, x+y > && \text{Definition of norm} \\
&= < x, x+y > + < y, x+y > && \text{Linearity in the first argument} \\
&= \overline{< x+y, x >} + \overline{< x+y, y >} && \text{Symmetry} \\
&= \overline{< x, x >} + \overline{< y, x >} + \overline{< x, y >} + \overline{< y, y >} && \text{Linearity in the first argument} \\
&= < x, x > + < x, y > + < y, x > + < y, y > && \text{Symmetry} \\
&= < x, x > + < y, y > && \text{Since } x \perp y \text{, i.e., } < x, y > = < y, x > = 0 \\
&= ||x||^2 + ||y||^2 && \text{Definition of norm} \quad \square
\end{aligned}$$

(13.10) Let $h = \prod_{i \in S} x_i$ be a monomial of degree $d = |S| \leq n - 1$, and let a be a 0-1 vector with at least $d + 1$ ones. There are only two possibilities:

(i) $a_i = 0$ for some $i \in S$. In this case, $h(b) = 0$ for all $b \leq a$ (trivially).

(ii) $a_i = 1$ for all $i \in S$. Let us define $B = \{b \in \mathbb{F}^n : b \leq a, a_i = 1, \forall i \in S\}$, i.e., B contains all vectors below a but fixing all coordinates in S to be one. It suffices to show that $|B|$ is an even number to show the result. Indeed, let k be the number of 1s other that the 1s fixed by a_i for $i \in S$. Since the total number of 1s is $d + 1$, we know that $k \geq 1$, i.e., there is at least one 1 in $a_j = 1$ for some $j \notin S$. Therefore, for each one of these 1s (outside of S) we can switch them to 0 to obtain a vector b such that $b \leq a$. There are 2^k ways of doing these. Hence,

$$\sum_{b \in B} h(b) = 2^k \cdot 1 \equiv 0 \pmod 2$$

Since both cases (i) and (ii) cover all possibilities, we can conclude that $\sum\limits_{b \leq a} h(b) = 0$.