M311
FINAL EXAM
SUMMER 2010

NAME: Enrique Areyan

(1) Let $f(x,y) = x^2 - \frac{1}{2}x^4 + y^2$

(a) Find the linear approximation to f at $(1,1)$.

Let $g(x,y,z) = x^2 - \frac{1}{2}x^4 + y^2 - z$. We know that the gradient of g is perpendicular to its level surfaces, i.e., perpendicular to f.

$\nabla g(x,y,z) = \langle 2x-2x^3, 2y, -1\rangle$. Hence, the normal vector for the tangent plane of $f(x,y)$ at $(1,1)$ is: $\nabla g(1,1,z) = \langle 0,2,-1\rangle = \vec{n}$.

the plane satisfies: $\vec{n}\cdot\langle x-1, y-1, z-f(1,1)\rangle = 0 \iff \langle 0,2,-1\rangle\cdot\langle x-1,y-1,z-\frac{3}{2}\rangle = 0$

Since, $f(1,1) = 1-\frac{1}{2}+1 = 2-\frac{1}{2} = \frac{3}{2}$ $\iff 2y-2+\frac{3}{2}-z=0 \iff 2y-\frac{1}{2}-z=0 \iff \boxed{2y-z=\frac{1}{2}}$

(b) At $(2,2)$, find the direction f is increasing fastest, and the rate of increase.

the direction of fastest increase is given by the gradient. Hence, at $(2,2)$, the direction is: $\nabla f(2,2) = \langle 2x-2x^3, 2y\rangle = \langle 2(2)-2(2)^3, 2(2)\rangle$
$= \langle 4-16, 4\rangle$
$= \langle -12, 4\rangle$

the rate of increase is $|\nabla f(2,2)| = |\langle -12,4\rangle|$
$= \sqrt{12^2+4^2} = \sqrt{144+16} = \sqrt{160}$

(c) Find the critical points of f, and use the 2nd derivative test to classify them.

the critical points satisfy: $\nabla f(x_0,y_0) = \vec{0}$. So we solve this eq:

$\nabla f(x_0,y_0) = \langle 2x_0-2x_0^3, 2y_0\rangle = \langle 0,0\rangle \iff 2x_0-2x_0^3=0$ and $2y_0=0$

$\iff x_0 = x_0^3$ and $y_0=0 \iff x_0 = 0, 1$ or -1 and $y_0=0$.

the critical points (x_0,y_0) are: $\boxed{(0,0), (1,0), (-1,0)}$

To classify these points we need to compute $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{vmatrix} = \begin{vmatrix} 2-6x_0^2 & 0 \\ 0 & 2\end{vmatrix}$

$= (2-6x_0^2)\cdot 2 - 0 = 4-12x_0^2$. For each critical point:

$(0,0)$: $D(0,0) = 4 > 0$ and $f_{xx}(0,0) = 2 > 0 \Rightarrow (0,0)$ is a $\boxed{\text{local min.}}$

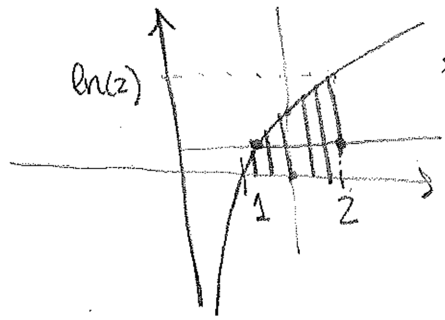
$(1,0)$: $D(1,0) = -8 < 0 \Rightarrow (1,0)$ is a $\boxed{\text{Saddle point}}$

$(-1,0)$: $D(-1,0) = -8 < 0 \Rightarrow (-1,0)$ is a $\boxed{\text{Saddle point.}}$

② Rewrite $\int_1^2 \int_0^{\ln x} f(x,y)\,dy\,dx$ to integrate in the opposite order ($dx\,dy$).

the Domain of integration D is:

D



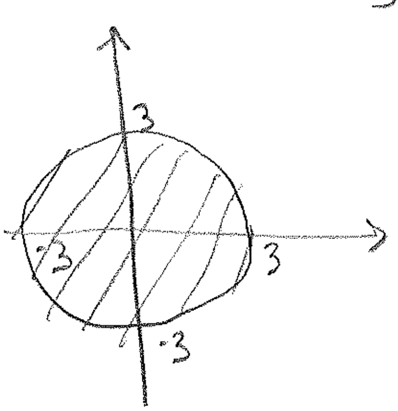
$$\int_1^2 \int_0^{\ln(x)} f(x,y)\,dy\,dx = \int_0^{\ln(2)} \int_{e^y}^{2} f(x,y)\,dx\,dy$$

③ Rewrite $\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^5 dz\,dy\,dx$ to integrate using cylindrical coordinates, and evaluate.

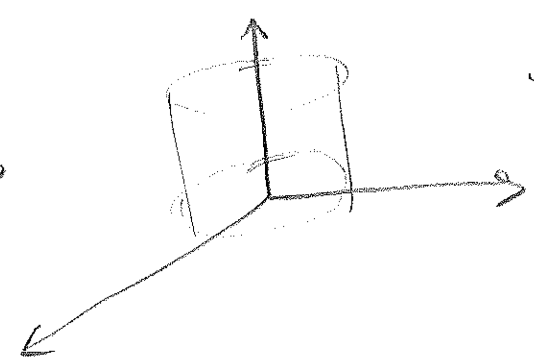
For cylindrical coordinates we use: $x = r\cos\theta$, $y = r\sin\theta$; $z = z$

$r^2 = x^2 + y^2$

the Domain D is



the solid E is



the cylindrical coordinates integral is:

$$\int_0^{2\pi} \int_0^{3} \int_1^{5} r\,dz\,dr\,d\theta$$

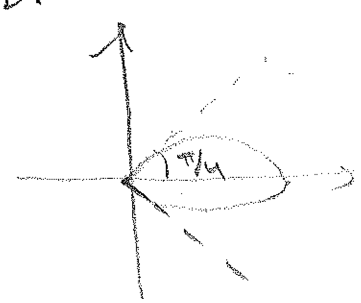
(4) Find the average value of $f(x,y,z) = xyz$ in the cube $0 \le x, y, z \le L$.

$$\text{Average} = \frac{\iiint_E f(x,y,z)\,dV}{\text{Volume of } E} = \frac{\int_0^L\int_0^L\int_0^L xyz\,dx\,dy\,dz}{L^3}$$

$$= \frac{\left[\frac{x^2}{2}\right]_0^L \left[\frac{y^2}{2}\right]_0^L \left[\frac{z^2}{2}\right]_0^L}{L^3} = \frac{\frac{L^2 \cdot L^2 \cdot L^2}{8}}{L^3} = \frac{L^6}{8L^3} = \boxed{\frac{L^3}{8}}$$

(5) Find the area enclosed by one "leaf" of the curve $r = \cos 2\theta$.

D.



$$\int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} 1 \cdot r\,dr\,d\theta = \int_{-\pi/4}^{\pi/4} \left[\frac{r^2}{2}\right]_0^{\cos 2\theta} d\theta$$

$r = \cos 2\theta = 0$

$\Rightarrow \cos 2\theta = 0$

$\cos 2\frac{\pi}{4} = \cos\frac{\pi}{2} = 0$

$\Rightarrow \boxed{\theta = \frac{\pi}{4}}$

$$= \int_{-\pi/4}^{\pi/4} \frac{\cos^2(2\theta)}{2}\,d\theta \; ; \quad u = 2\theta \; ; \; du = 2\,d\theta \Rightarrow d\theta = \frac{du}{2}$$

$$= \int_0^0 \frac{\cos^2(u)}{4}\,du$$

(b) Let $r(t) = \langle \cos t, \sin t, t \rangle$. $x(t) = \cos t$
$y(t) = \sin t$
$z(t) = t$

(a) Find the length of the curve, for $0 \le t \le 2\pi$.

$$\int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\, dt$$

$$= \int_0^{2\pi} \sqrt{2}\, dt = \boxed{2\pi\sqrt{2}}$$

(b) Find the tangent line to the curve, when $t = \pi$.

the tangent line has direction $\vec{r}'(t=\pi) = \langle -\sin t, \cos t, 1 \rangle = \langle 0, -1, 1 \rangle$

And passes through the point $\vec{P_0} = \vec{r}(t=\pi) = \langle \cos \pi, \sin \pi, 1 \rangle = \langle -1, 0, 1 \rangle$.

Hence, the eq. of the line is: $\vec{l}(t) = \vec{P_0} + t\vec{r}'(t=\pi) = \langle -1, 0, 1 \rangle + t\langle 0, -1, 1 \rangle$
$= \langle -1, -t, 1+t \rangle$

(c) Find T, N, and $B(t)$.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle -\sin t, \cos t, 1 \rangle}{\sqrt{\sin^2 t + \cos^2 t + 1}} = \boxed{\tfrac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle}$$

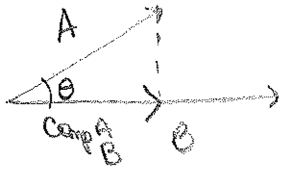
$$\vec{N}(t) = \frac{T'(t)}{|T'(t)|} = \frac{\langle -\cos t, -\sin t, 0 \rangle \frac{1}{\sqrt{2}}}{\sqrt{\frac{\cos^2 + \sin^2}{2}}} = \frac{\frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle}{\frac{1}{\sqrt{2}}} = \boxed{\langle -\cos t, -\sin t, 0 \rangle}$$

(d) Find the curvature $K(t)$.

$$\boxed{\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)}$$

7. Let $A = (3,1,-1)$ and $B = (1,1,1)$.

(a) What is the length of the projection of A onto B?



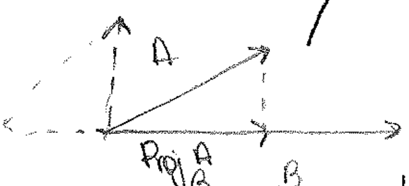
$$\cos\theta = \frac{\text{Comp}_B^A}{|A|} \Rightarrow |A|\cdot\cos\theta = \text{Comp}_B^A$$

$$A\cdot B = |A|\cdot|B|\cdot\cos\theta$$

$$\therefore A\cdot B = |B|\cdot\text{Comp}_B^A \iff \text{Comp}_B^A = \frac{A\cdot B}{|B|}$$

Hence, $\text{Comp}_B^A = \dfrac{\langle 3,1,-1\rangle\cdot\langle 1,1,1\rangle}{\sqrt{1^2+1^2+1^2}} = \dfrac{3+1-1}{\sqrt{3}} = \boxed{\dfrac{3}{\sqrt{3}}}$

(b) Write A as the sum of a vector parallel to B plus a vector orthogonal to B.



We want to find: $\vec{A} = \underbrace{\left(\vec{A} - \text{Proj}_B^A\right)}_{\text{orthogonal to } B} + \underbrace{\text{Proj}_B^A}_{\text{parallel to } B}$

where $\text{Proj}_B^A = \text{Comp}_B^A \cdot \dfrac{\vec{B}}{|\vec{B}|} = \dfrac{3}{\sqrt{3}}\cdot\dfrac{\langle 1,1,1\rangle}{\sqrt{3}} = \boxed{\langle 1,1,1\rangle}$

Hence, $\vec{A} = \langle 3,1,-1\rangle - \langle 1,1,1\rangle + \langle 1,1,1\rangle$
$= \langle 2,0,-2\rangle + \langle 1,1,1\rangle$, where $\langle 2,0,-2\rangle \perp \langle 1,1,1\rangle$, and $\langle 1,1,1\rangle = 1\cdot\langle 1,1,1\rangle$

(c) What is the area of the parallelogram spanned by A and B?

$$\text{Area} = |\vec{A}\times\vec{B}| = \left|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix}\right| = |\hat{i}(2) - \hat{j}(4) + \hat{k}(2)| = |\langle 2,-4,2\rangle|$$

$$= \sqrt{2^2+4^2+2^2} = \sqrt{16+4+4} = \sqrt{24} = \sqrt{6\cdot 4} = \sqrt{3\cdot 8} = \boxed{2\sqrt{6}}$$

8. Find the distance between the point $(1,2,3)$ and the line $r(t) = \langle 4,3,2\rangle + t\langle 1,1,1\rangle$.

We want to minimize the function:

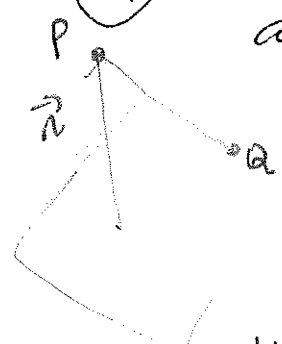
$$d(P,L)^2 = (4+t-1)^2+(3+t-2)^2+(2+t-3)^2$$
$$= (t+3)^2+(t+1)^2+(t-1)^2$$
$$= t^2+6t+9+t^2+2t+1+t^2-2t+1$$
$$= \boxed{3t^2+6t+11}$$

Derivative test:

$f'(t) = 6t+6 = 0 \Rightarrow \boxed{t=-1}$; $f''(t) = 6 > 0 \Rightarrow t=-1$ is the global min.

If $t=-1 \Rightarrow \vec{r}(-1) = \langle 3,2,1\rangle$. So this is the closest point from the line to the given point. Its distance is $d((1,2,3),(3,2,1)) = \sqrt{2^2+2^2} = \sqrt{8} = \boxed{2\sqrt{2}}$

9. Find the distance between the point $(1,2,3)$ and the plane $2x+3y+4z=9$.



Let Q be a point on the plane, say $(1,1,1)$. then $\vec{PQ} = (1,1,1)-(1,2,3) = (0,-1,-2)$. Now, we want to project $(0,-1,-2)$, onto $\vec{n}$, which we know is $(2,3,4) = \vec{n}$.

$$\mathrm{Comp}_{\vec{n}}\vec{PQ} = \frac{\vec{PQ}\cdot\vec{n}}{|\vec{n}|} = \frac{(0,-1,-2)\cdot(2,3,4)}{\sqrt{4+9+16}} = \frac{-2-8}{\sqrt{29}} = \frac{-11}{\sqrt{29}}$$

Since the distance is always positive, take:

$$\text{distance} = \left|\mathrm{Comp}_{\vec{n}}\vec{PQ}\right| = \left|\frac{-11}{\sqrt{29}}\right| = \boxed{\frac{11}{\sqrt{29}}}$$