Spring 2004 Math 253/501–503
12 Multivariable Differential Calculus
12.2 Limits and Continuity
Tue, 03/Feb ©2004, Art Belmonte

Summary

Let $\mathbf{a} \in \mathbb{R}^n$ and r be positive. The open n-ball $B(\mathbf{a}; r)$ of radius r centered at $\mathbf{a}$ is the set of all points in $\mathbb{R}^n$ whose distance from $\mathbf{a}$ is less than r. In other words, $B(\mathbf{a}; r) = \left\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| < r\right\}$. For $n = 2$ we have a circular disk, whereas for $n = 3$ we have a spherical ball.

With $\mathbf{x} = [x_1, \ldots, x_n]$, let $f(\mathbf{x})$be a real-valued function of n variables defined on a subset D of $\mathbb{R}^n$. Suppose that $\mathbf{a} \in \mathbb{R}^n$ is such that every open ball centered at $\mathbf{a}$ contains a point of D distinct from $\mathbf{a}$. Then the **limit** of $f(\mathbf{x})$ as $\mathbf{x}$ approaces $\mathbf{a}$ equals b if and only for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\| f(\mathbf{x}) - b\| < \epsilon$ whenever $\mathbf{x} \in D$ and $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$.

We write $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = b$ when the limit exists. Colloquially, the limit exists provided we can make $f(\mathbf{x})$ arbitrarily close to b by taking points in D sufficiently close to, but distinct from, $\mathbf{a}$.

If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$, we say that f is continuous at $\mathbf{a}$. Three criteria are required for continuity.

1. f is defined at $\mathbf{a}$; i.e., $\mathbf{a} \in D$, the domain of f.
2. The limit $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ exists.
3. This limit equals the function value $f(\mathbf{a})$.

If f is continuous at every point of a subset E of $\mathbb{R}^n$, we say that f is continuous on E.

Theorem Compositions of continuous functions are continuous.

NOTE Computer algebra systems have very limited capabilities when it comes to multivariable limits. This includes MATLAB's Symbolic Math Toolbox as well as the TI-89. Accordingly, analytical work needs to be done by hand. That said, the 3-D graphing capability of MATLAB can help to determine whether a limit exists at a point.

Hand Examples

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Find the limit, if it exists, or show why it does not exist.

$$\lim_{(x,y)\to(-3,4)} \left(x^3 + 3x^2y^2 - 5y^3 + 1\right)$$

Solution

Just as in Calc 1, if you can substitute numbers for variables and the expression is defined, then the numerical result *is* the limit. In our case, the limit is $(-3)^3 + 3(-3)^2(4)^2 - 5(4)^3 + 1 = 86$.

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Find the limit, if it exists, or show why it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}$$

Solution

- As $(x, y) \to (0, 0)$ along the x-axis ($y = 0$), we have $\dfrac{x^2}{x^2 + y^2} = \dfrac{x^2}{x^2} = 1 \to 1$.
- As $(x, y) \to (0, 0)$ along the line $y = x$, we have $\dfrac{x^2}{x^2 + y^2} = \dfrac{x^2}{2x^2} = \dfrac{1}{2} \to \dfrac{1}{2}$.
- Since these directional limits differ, we conclude that the limit $\lim_{(x,y)\to(0,0)} \dfrac{x^2}{x^2 + y^2}$ does *not* exist.

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Find the limit, if it exists, or show why it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$$

Solution

Observe that

$$0 \le \left|\frac{xy}{\sqrt{x^2 + y^2}}\right| \le \left|\frac{xy}{\sqrt{x^2}}\right| = \frac{|x|\,|y|}{|x|} = |y| \to 0 \text{ as } (x, y) \to (0, 0).$$

Therefore $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2 + y^2}} = 0$ via the Squeeze Theorem.

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Find the limit, if it exists, or show why it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$$

Solution

Rationalize the denominator, then plug-and-chug.

$$\begin{aligned}
&\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} \\
=&\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}\cdot\frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1} \\
=&\lim_{(x,y)\to(0,0)} \frac{\left(x^2+y^2\right)\left(\sqrt{x^2+y^2+1}+1\right)}{\left(x^2+y^2\right)} \\
=&\lim_{(x,y)\to(0,0)} \left(\sqrt{x^2+y^2+1}+1\right) = 2
\end{aligned}$$

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Find the limit, if it exists, or show why it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$$

Solution

- As $(x,y) \to (0,0)$ along the parabola $y = x^2$, we have $\left|\frac{xy^3}{x^2+y^6}\right| = \left|\frac{x^7}{x^2+x^{12}}\right| \le \frac{|x|^7}{|x|^2} = |x|^5 \to 0$. So the limit along this path is 0.
- As $(x,y) \to (0,0)$ along the cubic $x = y^3$, we have $\frac{xy^3}{x^2+y^6} = \frac{y^6}{2y^6} = \frac{1}{2} \to \frac{1}{2}$. So the limit along this second path is $\frac{1}{2}$.
- Since these directional limits differ, we conclude that the limit $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ does *not* exist.

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Let $g(t) = \frac{\sqrt{t}-1}{\sqrt{t}+1}$ and $f(x,y) = x^2 - y$. Find the composition $h(x,y) = g(f(x,y))$ and the set on which h is continuous.

Solution

- The composition is $h(x,y) = \frac{\sqrt{x^2-y}-1}{\sqrt{x^2-y}+1}$.
- Now h is defined provided $x^2 - y \ge 0$ or $y \le x^2$. Moreover, h is continuous on this domain, $D = \left\{(x,y) : y \le x^2\right\}$, the set of all points in the xy-plane on and below the parabola $y = x^2$.

Example A

Find the limit, if it exists, or show why it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2}$$

Solution

Switch to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$. Then, as $(x,y) \to (0,0)$, we have $r \to 0^+$ and hence $w = r^2 \to 0^+$. Thus $\lim_{(x,y)\to(0,0)} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2} = \lim_{r\to0^+} \frac{\sin\left(r^2\right)}{r^2} = \lim_{w\to0^+} \frac{\sin w}{w} = 1.$

MATLAB Examples

s739x28 [739/28 revisited]

Find the limit, if it exists, or show why it does not exist.

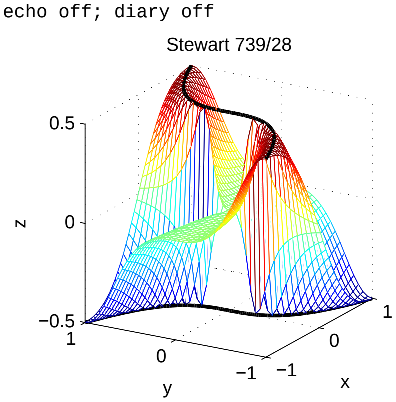
$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$$

Solution

You can see the different directional limits from this graph. Notice the "ridge" in the $z = \frac{1}{2}$ plane along the approach $x = y^3$. There's another ridge in the $z = -\frac{1}{2}$ plane along the approach $x = -y^3$.

Accordingly, $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ does *not* exist.

```
%
% Stewart 739/28
%
x = linspace(-1, 1, 40); y = x;
[X,Y] = meshgrid(x,y);
Z = X.*Y.^3 ./ (X.^2 + Y.^6);
mesh(X,Y,Z); hold on
t = linspace(-1, 1);
y = t; x = y.^3; z = 0.5 * ones(size(t));
plot3(x,y,z, 'k', 'LineWidth', 2)
y = t; x = -y.^3; z = -0.5 * ones(size(t));
plot3(x,y,z, 'k', 'LineWidth', 2)
%
echo off; diary off
```



Example A [revisited]

Find the limit, if it exists, or show why it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2}$$

Solution

Here we'll use polar coordinates. Unlike the preceding problem, the graph in this one levels off at $z = 1$—irrespective of the direction of approach. Therefore, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2} = 1$.

```
%
% Stewart 12.2/Example A
%
r = linspace(0, 4, 40);
t = linspace(0, 2*pi, 37);
[R,T] = meshgrid(r,t);
X = R .* cos(T);
Y = R .* sin(T);
Z = sin(R.^2) ./ R.^2;
surf(X,Y,Z)
%
echo off; diary off
```