Spring 2004 Math 253/501–503
12 Multivariable Differential Calculus
12.7 Maximum and Minimum Values
Tue, 17/Feb ©2004, Art Belmonte

Summary

With $\mathbf{x} = [x_1, \ldots, x_n]$, let $f(\mathbf{x})$ be a real-valued function of n variables defined on a subset D of $\mathbb{R}^n$.

- **local maximum of f at $\mathbf{x}_*$**: $f(\mathbf{x}_*) \geq f(\mathbf{x})$ whenever $\mathbf{x} \in B(\mathbf{x}_*; r) \cap D$ for some ball about $\mathbf{x}_*$; colloquially, $f(\mathbf{x}_*)$ is the "highest" point locally
- **absolute maximum of f at $\mathbf{x}_*$**: $f(\mathbf{x}_*) \geq f(\mathbf{x})$ for all $\mathbf{x} \in D$; colloquially, $f(\mathbf{x}_*)$ is the "highest" point globally
- **local minimum of f at $\mathbf{x}_*$**: $f(\mathbf{x}_*) \leq f(\mathbf{x})$ whenever $\mathbf{x} \in B(\mathbf{x}_*; r) \cap D$ for some ball about $\mathbf{x}_*$; colloquially, $f(\mathbf{x}_*)$ is the "lowest" point locally
- **absolute minimum of f at $\mathbf{x}_*$**: $f(\mathbf{x}_*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in D$; colloquially, $f(\mathbf{x}_*)$ is the "lowest" point globally
- A **critial point** (or **stationary point**) $\mathbf{x}_*$ is one for which $\vec{\nabla} f(\mathbf{x}_*) = \mathbf{0}$, the zero vector, or for which $\vec{\nabla} f(\mathbf{x}_*)$ does not exist (due to the fact that a partial derivative of f does not exist at $\mathbf{x}_*$).
- A local **extremum** is either a local maximum or local minimum. The plurals of maximum, minimum, and extremum are maxima, minima, and extrema, respectively.
- Recall that the **gradient vector** of f is

$$\vec{\nabla} f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right],$$

  a vector of first-order partial derivatives; in TAMUCALC, use the **grad** command.
- The **Hessian matrix** of f is $H = \begin{bmatrix} f_{x_1x_1} & \cdots & f_{x_1x_n} \\ \vdots & \ddots & \vdots \\ f_{x_nx_1} & \cdots & f_{x_nx_n} \end{bmatrix}$, a matrix of second-order partial derivatives; in TAMUCALC, use the **hess** command.
- The **LPMD**s (Leading Principal Minor Determinants) of the Hessian H are the determinants of the upper left square submatrices of H. We form a collection of them—the 1×1 determinant, the 2×2 determinant, etc.

Theorems

- If f has a local extremum at $\mathbf{x}_*$, then $\vec{\nabla} f(\mathbf{x}_*) = \mathbf{0}$, provided the first-order partial derivatives of f exist at $\mathbf{x}_*$.
- **EVT** (Extreme Value Theorem): If f is continuous on a closed bounded set D in $\mathbf{R}^n$, then f attains an absolute maximum value and absolute minimum value at some points in D.

Tests

SDT (Second Derivatives Test) Let $\mathbf{x}_*$ be a critical point of f with $\vec{\nabla} f(\mathbf{x}_*) = \mathbf{0}$ and assume that the Hessian of f is continuous in a neighborhood of $\mathbf{x}_*$. Consider the LPMDs at $\mathbf{x}_*$.

- LPMDs all positive $\Rightarrow$ local minimum of f at $\mathbf{x}_*$.
- LPMDs alternate in sign, starting with negative $(-, +, -, +, \ldots) \Rightarrow$ local maximum of f at $\mathbf{x}_*$.
- If neither #1 or #2 hold, then *in general* the Second Derivatives Test is inconclusive*.
- *EXCEPT, for a two-variable problem (the typical case for you!), if the second LPMD is negative, then f has a **saddle point** at x_*.

Extreme Values Test To find the absolute maximum and minimum values of a continuous function f on a closed bounded set D, proceed as follows.

1. Find the critical points of f in the *interior* of D (just solve $\vec{\nabla} f = \mathbf{0}$; you do NOT need to classify said points using the SDT); a Calc 3 problem.
2. Find the critical points of f on the *boundary* of D; typically Calc 1 or high school problems.
3. Crank out function values of f at the points you found in (a) and (b). The biggest is the absolute maximum; the smallest is the absolute minimum.

"Hand" Examples

The time has come for machine power! Use your TI-89 and TAMUCALC when doing problems "by hand."

781/6

Find all local extrema and saddle points of the function

$$f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2.$$

Solution

- Solve $\vec{\nabla} f = \left[6x^2 + 10x + y^2, \quad 2xy + 2y\right] = [0, 0]$ for x and y to obtain critical points $[x, y]$:

$$[0, 0], \quad \left[-\tfrac{5}{3}, 0\right], \quad [-1, 2], \quad [-1, -2].$$

- Compute the Hessian matrix

$$H(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 12x + 10 & 2y \\ 2y & 2x + 2 \end{bmatrix}$$

  and leading principal minor determinants (LPMDs)

$$L(x, y) = \left[12x + 10, \quad (12x + 10)(2x + 2) - 4y^2\right].$$

- Here is a table analyzing the four critical points. Verify the cell entries with your TI-89 as we did in class.

(x, y)	$f(x, y)$	$H(x, y)$	LPMDs	Classification
$(0, 0)$	0	$\begin{bmatrix} 10 & 0 \\ 0 & 2 \end{bmatrix}$	$[10, 20]$	local minimum
$\left(-\frac{5}{3}, 0\right)$	$\frac{125}{27} \approx 4.63$	$\begin{bmatrix} -10 & 0 \\ 0 & -\frac{4}{3} \end{bmatrix}$	$\left[-10, \frac{40}{3}\right]$	local maximum
$(-1, 2)$	3	$\begin{bmatrix} -2 & 4 \\ 4 & 0 \end{bmatrix}$	$[-2, -16]$	saddle point
$(-1, -2)$	3	$\begin{bmatrix} -2 & -4 \\ -4 & 0 \end{bmatrix}$	$[-2, -16]$	saddle point

- Of course, the pictures tell the story! See the corresponding MATLAB example for graphical verification of these assertions.

782/30

Find the absolute maxiumum and minimum values of the function $f(x, y) = 2x^2 + x + y^2 - 2$ on $D = \left\{(x, y) : x^2 + y^2 \le 4\right\}$.

Solution

The Extreme Value Theorem guarantees that f indeed attains maximum and minimum values on the closed circular disk D.

- For the interior of D, solve $\vec{\nabla} f = [4x + 1, 2y] = [0, 0]$ to obtain $(x, y) = \left(-\frac{1}{4}, 0\right)$, which *is* in D. (If it were not in D, we'd toss this point out.)

- For the boundary of D, substitute $x^2 + y^2 = 4$ into $f(x, y)$ to obtain $g(x) = x^2 + x + 2, \ -2 \le x \le 2$. Solve $g'(x) = 2x + 1 = 0$ to obtain $x = -\frac{1}{2}$, which is in the open interval $(-2, 2)$. (Were it not, we'd toss it out.) For this value of x, we have $y = \pm\sqrt{4 - x^2} = \pm\sqrt{15/4} = \pm\frac{1}{2}\sqrt{15}$.

- Don't forget to check the *boundary* of the boundary; i.e., the endpoints of the interval $[-2, 2]$! When $x = \pm 2$, we have $y = 0$.

- Crank out function values of f at the points encountered.

(x, y)	$f(x, y)$	Comments
$\left(-\frac{1}{4}, 0\right)$	$-\frac{17}{8} = -2.125$	absolute mininum on D
$\left(-\frac{1}{2}, -\frac{1}{2}\sqrt{15}\right)$	$\frac{7}{4} = 1.75$	(intermediate value)
$\left(-\frac{1}{2}, \frac{1}{2}\sqrt{15}\right)$	$\frac{7}{4} = 1.75$	(intermediate value)
$(-2, 0)$	4	(intermediate value)
$(2, 0)$	8	absolute maximum on D

- See the corresponding MATLAB example for a nice **surfc** (surface *and* contour) pic of this surface.

782/48

Find the dimensions of the rectangular box with largest volume if the total surface area is given as 64 cm^2.

Solution

We'll do this one together on our TI-89s.

Draw a diagram. Let x, y, and z be the length, width, and height of the box, respectively. The volume of the box is $V = xyz$, whereas its surface area is $S = 2xy + 2yz + 2xz = 64$. Solving the latter for z yields $z = \dfrac{32 - xy}{x + y}$. Hence $f(x, y) = V(x, y, z) = \dfrac{xy(32 - xy)}{x + y}$ via substitution.

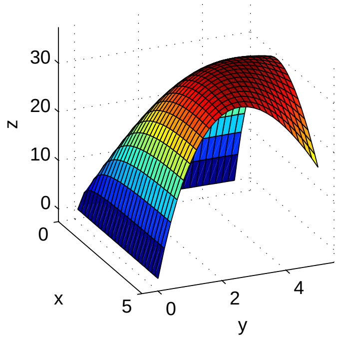
- Solve $\vec{\nabla} f = \mathbf{0}$ to obtain $x = y = \frac{4}{3}\sqrt{6} \approx 3.27$ cm. ("Use the Force, Luke. . .")

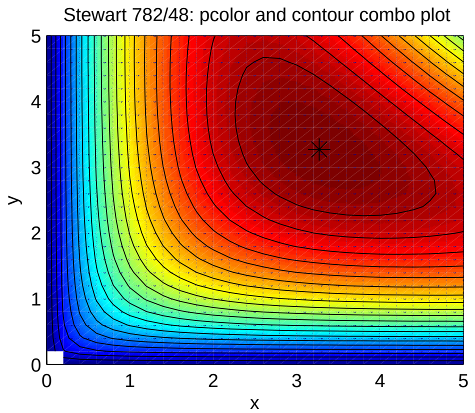
- Physically this must give the solution, whence $z = \frac{4}{3}\sqrt{6} \approx 3.27$ cm and $V = \frac{128}{9}\sqrt{6} \approx 34.84 \text{ cm}^3$.

- You may check, however, that at $(x, y) = \left(\frac{4}{3}\sqrt{6}, \frac{4}{3}\sqrt{6}\right)$ the LPMDs are $\left\{-\frac{4}{3}\sqrt{6}, 8\right\}$ or $\{-, +\}$, signifying a [local] maximum.

- More evidence is had from the fact that along the "boundary" $x = 0$, we have $V = 0$. Similarly, $V = 0$ along $y = 0$. Here are illustrative graphs!

Stewart 782/48: Volume of box as a function of x and y





MATLAB Examples

s781x06 [781/6 revisited]

Find all local extrema and saddle points of the function

$$f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2.$$

Solution

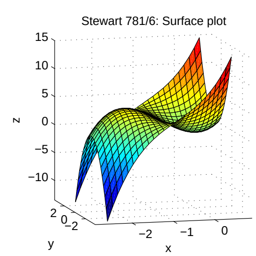
Here we replicate the symbolic work we did with our TI-89.

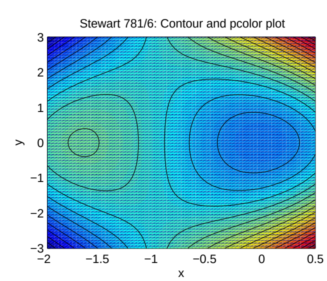
```
% Stewart 781/6: symbolic work
%
syms x y; v = [x y];
f = 2*x^3 + x*y^2 + 5*x^2 + y^2;
pretty(f)

                                          3       2       2     2
                                       2 x  + x y  + 5 x  + y
g = grad(f,v); pretty(g)

                                  [   2     2                    ]
                                  [6 x  + y  + 10 x    2 x y + 2 y]
c = solve(g(1), g(2))
c =
    x: [4x1 sym]
    y: [4x1 sym]
c = [c.x c.y]; c
c =
[    0,     0]
[ -5/3,     0]
[   -1,     2]
[   -1,    -2]
%
h = Hess(f,v); L = LPMD(h);
pretty(h); pretty(L)

                                       [12 x + 10      2 y  ]
                                       [                    ]
                                       [    2 y       2 x + 2]

                               [                 2                  2]
                               [12 x + 10    24 x  + 44 x + 20 - 4 y ]
%
echo off
for k = 1:size(c,1)
    p = c(k,:)
    func_val = subs(f, [x y], p)
    Hessian = subs(h, [x y], p)
    LPMDs = subs(L, [x y], p)
end
```

```
p =
[ 0, 0]
func_val =
0
Hessian =
[ 10,  0]
[  0,  2]
LPMDs =
[ 10, 20]
p =
[ -5/3,    0]
func_val =
125/27
Hessian =
[   -10,    0]
[     0, -4/3]
LPMDs =
[   -10, 40/3]
p =
[ -1,  2]
func_val =
3
Hessian =
[ -2,  4]
[  4,  0]
LPMDs =
[  -2, -16]
p =
[ -1, -2]
func_val =
3
Hessian =
[ -2, -4]
[ -4,  0]
LPMDs =

[  -2, -16]
```

Now we illustrate the extrema and saddle points with surface and contour graphs!

```
%
% Stewart 781/6g
%
x = linspace(-2, 0.5, 25); y = linspace(-3, 3, 25);
[X,Y] = meshgrid(x,y);
Z = 2*X.^3 + X.*Y.^2 + 5*X.^2 + Y.^2;
surf(X,Y,Z); grid on
%
figure
x = linspace(-2, 0.5, 75); y = linspace(-3, 3, 75);
[X,Y] = meshgrid(x,y);
Z = 2*X.^3 + X.*Y.^2 + 5*X.^2 + Y.^2;
pcolor(X,Y,Z); shading interp
hold on; contour(X,Y,Z,20,'k')
%

echo off; diary off
```





s782x30 [782/30 revisited]

Find the absolute maxiumum and minimum values of the function $f(x,y) = 2x^2 + x + y^2 - 2$ on $D = \{(x,y) : x^2 + y^2 \le 4\}$.

Solution

Here is code which draws a combination surface/contour plot that is illustrative, followed by the graph.

```
%
% Stewart 782/30
%
r = linspace(0, 2, 21); t = linspace(0, 2*pi, 37);
[R,T] = meshgrid(r,t);
X = R .* cos(T); Y = R .* sin(T);
Z = 2*X.^2 + X + Y.^2 - 2;
surfc(X,Y,Z); grid on
%
echo off; diary off
```