Spring 2004 Math 253/501–503
12 Multivariable Differential Calculus
12.5 The Chain Rule
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Summary

- With $\mathbf{x} = [x_1, \ldots, x_n]$, let $u = f(\mathbf{x})$ be a real-valued function of n variables defined on a subset D of $\mathbb{R}^n$. In turn, $\mathbf{x}$ is a vector-valued funtion of m variables, defined on a subset E of $\mathbb{R}^m$ with $\mathbf{x}(E) \subset D$.

  $$\mathbf{x} = \mathbf{x}(\mathbf{t}) = [x_1(t_1, \ldots, t_m), \ldots, x_n(t_1, \ldots, t_m)]$$

  The [partial] derivative $\partial u/\partial t_j$ is given by

  $$\frac{\partial u}{\partial t_j} = \vec{\nabla} u \cdot \frac{\partial \mathbf{x}}{\partial t_j} = \left[\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right] \cdot \left[\frac{\partial x_1}{\partial t_j}, \ldots, \frac{\partial x_n}{\partial t_j}\right] = \sum_{k=1}^{n} \frac{\partial u}{\partial x_k}\frac{\partial x_k}{\partial t_j}.$$

- NOTE: If $m = 1$, then du/dt and $d\mathbf{x}/dt$ are ordinary derivatives (as opposed to partial derivatives). This is immaterial to the mechanical computation of derivatives.

- With $\mathbf{x} = [x_1, \ldots, x_n]$, suppose that $F(\mathbf{x}) = 0$ implicitly defines x_j as a function of the other x_k. Then a fast way to compute $\partial x_j/\partial x_k$ implicitly is

  $$\frac{\partial x_j}{\partial x_k} = -\frac{\partial F/\partial x_k}{\partial F/\partial x_j}, \text{ for } k \neq j.$$

Hand Examples

762/5

Given $w = xy^2z^3$, $x = \sin t$, $y = \cos t$, $z = 1 + e^{2t}$, use the Chain Rule to find dw/dt.

Solution

Let $\mathbf{g} = [x, y, z]$. Then

$$\begin{aligned} \frac{dw}{dt} &= \vec{\nabla} w \cdot \frac{d\mathbf{g}}{dt} \\ &= [w_x, w_y, w_z] \cdot \left[\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right] \\ &= \left[y^2z^3, 2xyz^3, 3xy^2z^2\right] \cdot \left[\cos t, -\sin t, 2e^{2t}\right] \\ &= y^2z^3\cos t - 2xyz^3\sin t + 6xy^2z^2e^{2t} \end{aligned}$$

or $\left(1+e^{2t}\right)^3\cos^3 t - 2\left(1+e^{2t}\right)^3\sin^2 t\cos t + 6e^{2t}\left(1+e^{2t}\right)^2\sin t\cos^2 t$ after substitution.

762/14

Write out the Chain Rule for the case that $w = f(x, y, z)$ and $x = x(t, u)$, $y = y(t, u)$, $z = z(t, u)$.

Solution

Let $\mathbf{g} = [x, y, z]$. Then

$$\begin{aligned} \frac{\partial w}{\partial t} &= \vec{\nabla} w \cdot \frac{\partial \mathbf{g}}{\partial t} \\ &= [f_x, f_y, f_z] \cdot \left[\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right] \\ &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}. \end{aligned}$$

The expansion of $\partial w/\partial u$ is similar.

763/28

Find $\partial z/\partial x$ and $\partial z/\partial y$ if $xyz = \cos(x + y + z)$.

Solution

Define $F(x, y, z) = xyz - \cos(x + y + z)$. Then $F(x, y, z) = 0$ implicitly defines z as a function of x and y. Hence

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{yz + \sin(x + y + z)}{xy + \sin(x + y + z)}.$$

Similarly, $\dfrac{\partial z}{\partial y} = -\dfrac{xz + \sin(x + y + z)}{xy + \sin(x + y + z)}$.

763/38

Car A is traveling north on Highway 16 at 90 km/h. Car B is traveling west on Highway 83 at 80 km/h. Each car is approaching the intersection of these highways. How fast is the distance between the cars changing when car A is 0.3 km from the intersection and car B is 0.4 km from the intersection?

Solution

Let x be the directed distance of car B from the intersection and y be the directed distance of car A from the intersection. Let z be the distance between the cars. (Draw a diagram!) By the Pythagorean Theorem, $z = \sqrt{x^2 + y^2}$. Let $\mathbf{g} = [x, y]$. Then

$$\begin{aligned} \frac{dz}{dt} &= \vec{\nabla} z \cdot \frac{d\mathbf{g}}{dt} \\ &= [z_x, z_y] \cdot \left[\frac{dx}{dt}, \frac{dy}{dt}\right] \end{aligned}$$

continued ⟶

$$
\begin{aligned}
&= \left[\tfrac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x),\, \tfrac{1}{2}\left(x^2+y^2\right)^{-1/2}(2y)\right]\cdot\left[\tfrac{dx}{dt},\, \tfrac{dy}{dt}\right] \\
&= \left[\frac{x}{\sqrt{x^2+y^2}},\, \frac{y}{\sqrt{x^2+y^2}}\right]\cdot\left[\frac{dx}{dt},\, \frac{dt}{dt}\right] \\
&= \left[\frac{0.4}{\sqrt{(0.4)^2+(-0.3)^2}},\, \frac{-0.3}{\sqrt{(0.4)^2+(-0.3)^2}}\right]\cdot[-80, 90] \\
&= -118\ \text{km/h}.
\end{aligned}
$$

MATLAB Examples

s762x10

Given $z = x\tan^{-1}(xy)$, $x = t^2$, $y = se^t$, use the Chain Rule to find $z_s = \partial z/\partial s$ and $z_t = \partial z/\partial t$.

Solution

Herewith the needful, with and without substitution. Symbolic variables in MATLAB are by default complex. Note the declaration that makes them **real** or **unreal** (complex). Tildes (~) appear when there are assumptions on variables (such as the fact that they're real and/or positive)—and only when using **pretty**.

```
%
% Stewart 762/10
%
syms s t x y real
z = x * atan(x*y);
grad_z = grad(z,[x,y]); pretty(grad_z) % Tildes!

                  [                                  2    ]
                  [                 x~ y~           x~    ]
                  [atan(x~ y~) + -----------    -----------]
                  [                    2   2          2   2]
                  [              1 + x~   y~    1 + x~   y~]
%
g = [t^2, s*exp(t)];
g_s = diff(g,s)

g_s =

[      0, exp(t)]

g_t = diff(g,t)

g_t =

[      2*t, s*exp(t)]

%
z_s_decaf = dot(grad_z, g_s);
z_t_decaf = dot(grad_z, g_t);
z_s_leaded = subs(z_s_decaf, [x y], g);
z_t_leaded = subs(z_t_decaf, [x y], g);
%
syms s t x y unreal % Ax those tildes!
pretty(z_s_decaf)  % z_s w/o  substitution

                                   2
                                  x  exp(t)
                                  ---------
                                      2  2
                                  1 + x   y
pretty(z_t_decaf)  % z_t w/o  substitution

                                                          2
                     /              x y   \     x  s exp(t)
                   2 |atan(x y) + ---------| t + -----------
                     |               2  2|          2  2
                     \          1 + x   y /      1 + x   y
pretty(z_s_leaded) % z_s with substitution

                                4
                               t  exp(t)
                            -----------------
                                4  2       2
                            1 + t   s  exp(t)
pretty(z_t_leaded) % z_t with substitution

/                               2               \
|         2                    t  s exp(t)      |
|2 atan(t  s exp(t)) + 2 -----------------| t
|                              4  2       2|
\                          1 + t   s  exp(t) /
           4
          t  s exp(t)
     + -----------------
            4  2       2
       1 + t   s  exp(t)
%
echo off; diary off
```

s763x18

Given $u = xy + yz + zx$, $x = st$, $y = e^{st}$, $z = t^2$, find $u_s = \partial u/\partial s$ and $u_t = \partial u/\partial t$ when $s = 0$ and $t = 1$.

Solution

Once the dust settles and substitutions are made, we see that $u_s(0, 1) = 3$ and $u_t(0, 1) = 2$.

```
%
%
% Stewart 763/18
%
syms s t x y z real
u = x*y + y*z + z*x;
grad_u = grad(u,[x,y,z])

grad_u =

[ y+z, x+z, y+x]

%
g = [s*t, exp(s*t), t^2];
g_s = diff(g,s)

g_s =

[          t, t*exp(s*t),          0]

g_t = diff(g,t)

g_t =

[          s, s*exp(s*t),        2*t]

%
u_s_decaf = dot(grad_u, g_s);
u_t_decaf = dot(grad_u, g_t);
u_s_leaded = subs(u_s_decaf, [x y z], g);
u_t_leaded = subs(u_t_decaf, [x y z], g);
%
syms s t x y unreal % Ax those tildes!
pretty(u_s_leaded) % u_s with substitution
```

```
                          2               2
               (exp(s t) + t ) t + (s t + t ) t exp(s t)
pretty(u_t_leaded) % u_t with substitution

              2               2
(exp(s t) + t ) s + (s t + t ) s exp(s t)
+ (2 exp(s t) + 2 s t) t
%
us10 = subs(u_s_leaded, [s t], [0 1])
us10 =
     3
ut10 = subs(u_t_leaded, [s t], [0 1])
ut10 =
     2
%
echo off; diary off
```

s763x28 [763/28 revisited]

Find $\partial z/\partial x$ and $\partial z/\partial y$ if $xyz = \cos(x + y + z)$.

Solution

Define $F(x, y, z) = xyz - \cos(x + y + z)$. Then $F(x, y, z) = 0$ implicitly defines z as a function of x and y. The **idiff** command I wrote then yields the requisite implicit derivatives.

```
%
% Stewart 763/28
%
syms x y z
F = x*y*z - cos(x+y+z);
z_x = idiff(F,z,x); pretty(z_x)

                             -y z - sin(y + x + z)
                             ---------------------
                             x y + sin(y + x + z)
z_y = idiff(F,z,y); pretty(z_y)

                             -z x - sin(y + x + z)
                             ---------------------
                             x y + sin(y + x + z)
%
echo off; diary off
```

s763x40

If $u = f(x, y)$, where $x = e^s \cos t$ and $y = e^s \sin t$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s}\left(\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right).$$

Solution

"Simply" show that left−right simplifies to zero. (There's actually quite a bit going on behind the scenes!)

```
%
% Stewart 763/40
%
syms s t u x y real
u = sym('f(x,y)');
grad_u = grad(u,[x,y])

grad_u =

[ diff(f(x,y),x), diff(f(x,y),y)]

%
g = [exp(s)*cos(t), exp(s)*sin(t)]

g =

[ exp(s)*cos(t), exp(s)*sin(t)]

g_s = diff(g,s)

g_s =

[ exp(s)*cos(t), exp(s)*sin(t)]

g_t = diff(g,t)

g_t =

[ -exp(s)*sin(t),  exp(s)*cos(t)]

%
u_s = grad_u * g_s.'

u_s =

diff(f(x,y),x)*exp(s)*cos(t)+diff(f(x,y),y)*exp(s)*sin(t)

u_t = grad_u * g_t.'

u_t =

-diff(f(x,y),x)*exp(s)*sin(t)+diff(f(x,y),y)*exp(s)*cos(t)

%
left = grad_u * grad_u.'; % sum of squares
v = [u_s, u_t];
right = exp(-2*s) * (v * v.');
It_is_zero = simple(left - right)

It_is_zero =

0

%
echo off; diary off
```