Spring 2004 Math 253/501–503
11 Three Dimensional Analytic Geometry and Vectors
11.2 Vectors and the Dot Product
Tue, 20/Jan ©2004, Art Belmonte

Summary

VECTORS: Terms and concepts

- **vector**: A quantity having magnitude (length) and direction.
  - *Geometrically*, an equivalence class of directed (hyper)line segments in n-D space having the same magnitude and direction (analogy: $\frac{1}{2}$ and $\frac{2}{4}$ are equivalent fractions). A particular member of this equivalence class has a point of application—the “tail” of the vector.
  - *Analytically*, an ordered n-tuple of (real) numbers, called components or elements: $\mathbf{a} = [a_1, a_2, \ldots, a_n]$. While Stewart uses angle brackets $\langle \ldots \rangle$ to delimit vectors, most folks (other authors, MATLAB, TI-89) use square brackets; so shall we! We also write $\mathbf{a}$ as $\vec{a}$.
- **position vector**: The distinguished member of the equivalence class that starts at the origin of n-D space.
- $\overrightarrow{AB}$: Vector from $A\,(a_1, \ldots, a_n)$ to $B\,(b_1, \ldots, b_n)$, realized as $[b_1 - a_1, \ldots, b_n - a_n]$; i.e., *end*−*start* in each slot.
- **magnitude**: The length of a (real) vector $\mathbf{a} = [a_1, \ldots, a_n]$ is

  $$\|\mathbf{a}\| = \sqrt{\sum_{k=1}^{n} a_k^2}$$

  via repeated application of the Pythagorean Theorem. Thus

  $$\left\|\overrightarrow{AB}\right\| = \sqrt{\sum_{k=1}^{n} (b_k - a_k)^2}.$$

- **zero vector**: This is the n-D vector all of whose components are zero: $\mathbf{0} = [0, 0, \ldots, 0]$. It has length zero and no specific direction.
- **vector addition / vector sum**: Add components slotwise.

  $$\mathbf{a} + \mathbf{b} = [a_1, \ldots, a_n] + [b_1, \ldots, b_n] = [a_1 + b_1, \ldots, a_n + b_n]$$

- **Triangle Law / Parallelogram Law**: The geometric (head-to-tail) interpretation of vector addition.
- **scalar**: a real number or symbol [more generally, a complex number or symbol or an element of a field]
- **scalar multiplication**: Given a scalar c and a vector $\mathbf{a}$, the scalar multiple of c with $\mathbf{a}$ is obtained by multiplying each component of $\mathbf{a}$ by c.

  $$c\mathbf{a} = c\,[a_1, \ldots, a_n] = [ca_1, \ldots, ca_n]$$

- **vector subtraction / vector difference**: Formally, $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b}$. Just subtract components slotwise.

  $$\mathbf{a} - \mathbf{b} = [a_1, \ldots, a_n] - [b_1, \ldots, b_n] = [a_1 - b_1, \ldots, a_n - b_n]$$

- **standard basis vectors**: In V_n (the set of all n-D vectors), these are the vectors $\mathbf{e}_1, \ldots \mathbf{e}_n$, where $\mathbf{e}_k$ has a 1 in the k^{th} slot and $n - 1$ zeros in its other slots. In V_3, we have the following aliases for $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, an **orthonormal** basis.

  $$\begin{aligned} \mathbf{i} &= \mathbf{e}_1 = [1, 0, 0] \\ \mathbf{j} &= \mathbf{e}_2 = [0, 1, 0] \\ \mathbf{k} &= \mathbf{e}_3 = [0, 0, 1]. \end{aligned}$$

  Note that we may write a given vector in terms of standard basis vectors. For example,

  $$\begin{aligned} \mathbf{a} &= [a_1, a_2, a_3] \\ &= [a_1, 0, 0] + [0, a_2, 0] + [0, 0, a_3] \\ &= a_1\,[1, 0, 0] + a_2\,[0, 1, 0] + a_3\,[0, 0, 1] \\ &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}. \end{aligned}$$

- **unit vector**: A vector whose length is 1. Given a nonzero vector $\mathbf{a} \neq \mathbf{0}$, the unit vector in the direction of $\mathbf{a}$ is $\mathbf{a}$-“hat.”

  $$\hat{\mathbf{a}} = \frac{1}{\|\mathbf{a}\|}\,\mathbf{a} = \frac{\mathbf{a}}{\|\mathbf{a}\|}.$$

- **resultant vector**: The vector sum of several vectors. For example, the resultant force is the vector sum of several forces. Again, the resultant velocity is the vector sum of several velocities.

VECTORS: Properties

In the following, c and d are scalars; $\mathbf{a}$ and $\mathbf{b}$ are vectors.

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (commutativity)
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ (associativity)
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$ (additive identity: the zero vector)
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ (additive inverse of $\mathbf{a}$: $-\mathbf{a}$)
5. $c\,(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$ (Scalar multiplication distributes over vector addition.)
6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$ (another instance of distributivity)
7. $(cd)\mathbf{a} = c(d\mathbf{a})$ (associativity of scalar multiples)
8. $1\mathbf{a} = \mathbf{a}$

DOT PRODUCT: Definitions and facts

Let $\mathbf{a} = [a_1, \ldots, a_n]$ and $\mathbf{b} = [b_1, \ldots, b_n]$ be n-D vectors.

- **dot product** (math definition): $\mathbf{a} \cdot \mathbf{b} = \sum_{k=1}^{n} a_k b_k$

- **dot product** (physics definition): $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta$, where θ is the angle between the vectors **a** and **b**; equivalent to math definition

- **angle between nonzero vectors**: $\theta = \cos^{-1}\left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}\right)$

- **orthogonality**: Vectors **a** and **b** are **perpendicular** if and only if $\mathbf{a}\cdot\mathbf{b} = 0$.

- **scalar projection** of **b** onto **a**: $\text{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$

- **vector projection** of **b** onto **a**: $\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\right)\dfrac{\mathbf{a}}{\|\mathbf{a}\|}$; i.e., the scalar projection times the unit vector in the direction of **a**; also known as the **parallel projection**

- **orthogonal projection** of **b** onto **a**: $\text{orth}_{\mathbf{a}}\mathbf{b} = \mathbf{b} - \text{proj}_{\mathbf{a}}\mathbf{b}$, whence **b** is the vector sum of a vector parallel to **a** and a vector perpendicular to **a**

- **work**: $W = \mathbf{F}\cdot\mathbf{d} = \|\mathbf{F}\|\,\|\mathbf{d}\|\cos\theta$, where **F** is the (constant) force vector and **d** is the displacement vector

- **direction cosines**: the components of $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{\|\mathbf{a}\|}$, the unit vector in the direction of **a**

- **direction angles**: the function $\cos^{-1} = \arccos$ mapped onto the direction cosine vector $\hat{\mathbf{a}}$; gives the angles said vector (and hence **a** itself) makes with the positive axes in $\mathbb{R}^n$

DOT PRODUCT: Properties

Here **a**, **b**, and **c** are vectors and k is a scalar.

1. $\mathbf{a}\cdot\mathbf{a} = \|\mathbf{a}\|^2$
2. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ (commutativity)
3. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$ (The dot product distributes over vector addition.)
4. $(k\mathbf{a})\cdot\mathbf{b} = k(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(k\mathbf{b})$ (Scalars roam freely across the dot product operation.)
5. $\mathbf{0}\cdot\mathbf{a} = 0$ (The dot product of zero vector with any other vector is zero.)

Hand Examples

664/10

Let $\mathbf{a} = 6\mathbf{i} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$. Find $\|\mathbf{a}\|$, $\mathbf{a}+\mathbf{b}$, $\mathbf{a}-\mathbf{b}$, $2\mathbf{a}$, and $3\mathbf{a}+4\mathbf{b}$.

Solution

Recall the discussion of standard basis vectors in the Summary. Rewrite the vectors as $\mathbf{a} = [6, 0, 1]$ and $\mathbf{b} = [1, -2, 7]$. (Make certain that you understand this.) We then have

$$\begin{aligned}
\|\mathbf{a}\| &= \sqrt{6^2+0^2+1^2} = \sqrt{37} \\
\mathbf{a}+\mathbf{b} &= [7, -2, 8] \\
\mathbf{a}-\mathbf{b} &= [5, 2, -6] \\
2\mathbf{a} &= [12, 0, 2] \\
3\mathbf{a}+4\mathbf{b} &= [18, 0, 3] + [4, -8, 28] = [22, -8, 31]\,.
\end{aligned}$$

Compare this with the corresponding MATLAB example. (Also remind me to show you on a TI-89.)

664/14

Find the unit vector that has the same direction as $\mathbf{a} = 2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$.

Solution

Rewrite **a** as $[2, -4, 7]$. Then

$$\hat{\mathbf{a}} = \frac{[2,-4,7]}{\sqrt{4+16+49}} = \left[\frac{2}{\sqrt{69}}, -\frac{4}{\sqrt{69}}, \frac{7}{\sqrt{69}}\right]$$

664/18

Given $\mathbf{a} = [-1, -2, -3]$ and $\mathbf{b} = [2, 8, -6]$, find the dot product $\mathbf{a}\cdot\mathbf{b}$.

Solution

We have

$$\mathbf{a}\cdot\mathbf{b} = (-1)(2) + (-2)(8) + (-3)(-6) = -2 - 16 + 18 = 0.$$

Therefore, **a** and **b** are orthogonal (perpendicular to one another)!

664/16

Find $\mathbf{a}\cdot\mathbf{b}$, given that $\|\mathbf{a}\| = 6$, $\|\mathbf{b}\| = \frac{1}{3}$, and the angle between **a** and **b** is $\theta = \frac{\pi}{4}$.

Solution

Via the physics definition of dot product we have

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta = (6)\left(\tfrac{1}{3}\right)\left(\tfrac{\sqrt{2}}{2}\right) = \sqrt{2}.$$

665/34

Find the values of x such that the vectors $\mathbf{v} = [x, x, -1]$ and $\mathbf{w} = [1, x, 6]$ are orthogonal.

Solution

For the given vectors to be orthogonal (perpendicular), their dot product must equal zero.

$$\begin{aligned} \mathbf{v}\cdot\mathbf{w} &= 0 \\ (x)(1)+(x)(x)+(-1)(6) &= 0 \\ x^2+x-6 &= 0 \\ (x-2)(x+3) &= 0 \\ x &= -3, 2 \end{aligned}$$

665/48

Find the scalar and vector projections of $\mathbf{b} = \mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$ onto $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

Solution

- The scalar projection is

$$\begin{aligned} \text{comp}_{\mathbf{a}}\mathbf{b} &= \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|} \\ &= \frac{2-18-2}{\sqrt{4+9+1}} \\ &= -\frac{18}{\sqrt{14}} \text{ or } -\frac{9\sqrt{14}}{7}. \end{aligned}$$

- The vector projection is

$$\begin{aligned} \text{proj}_{\mathbf{a}}\mathbf{b} &= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|} \\ &= -\frac{18}{\sqrt{14}}\,\frac{[2,-3,1]}{\sqrt{14}} \\ &= -\tfrac{9}{7}\,[2,-3,1] \\ &= \left[-\frac{18}{7}, \frac{27}{7}, -\frac{9}{7}\right]. \end{aligned}$$

665/54

Find the work done by a force of 20 lb acting in the direction N50°W in moving an object 4 ft due west.

Solution

The angle between the force and displacement vectors is 40°. The definition of work and our trusty TI-89 give

$$W = \mathbf{F}\cdot\mathbf{d} = \|\mathbf{F}\|\,\|\mathbf{d}\|\cos\theta = (20)\,(4)\,\left(\cos 40^\circ\right) \approx 61.28 \text{ ft-lb}.$$

MATLAB Examples

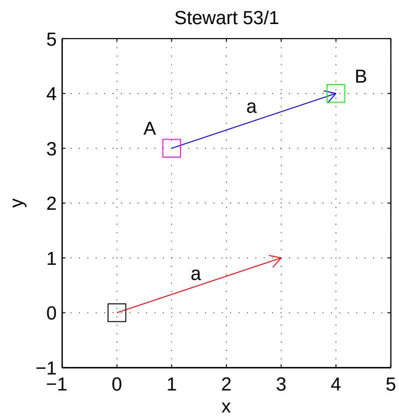
s053x01

Find a vector **a** with representation given by the directed line segment $\overrightarrow{AB}$ connecting points $A(1,3)$ and $B(4,4)$. Draw $\overrightarrow{AB}$ and the equivalent representation that starts at the origin.

Solution

We have $\mathbf{a} = \overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = [4,4]-[1,3]=[3,1]$. Here is a diary file and a diagram; **arrow** is a command Cooper wrote.

```
%
% Stewart 53/1
%
origin = [0 0]; A = [1 3]; B = [4 4];
a = B - A
a =
     3     1
arrow(A, a); hold on; axis equal
arrow(origin, a, 'r'); grid on
axis([-1 5 -1 5])
plot(A(1), A(2), 'ms', 'MarkerSize', 12);
plot(B(1), B(2), 'gs', 'MarkerSize', 12)
plot(0, 0, 'ks', 'MarkerSize', 12)
%
echo off; diary off
```



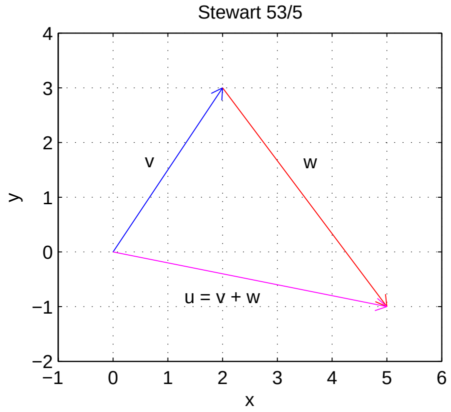
s053x05

Find the sum of the vectors $\mathbf{v} = [2,3]$ and $\mathbf{w} = [3,-4]$, then illustrate geometrically.

Solution

We have $\mathbf{v}+\mathbf{w} = [5,-1]$. Here is a diary file followed by a diagram that illustrates vector addition via the Triangle Law—the "head-to-tail" interpretation of vector addition.

```
%
% Stewart 53/5
%
v = [2 3]; w = [3 -4];
o = [0 0]; u = v + w
u =
     5    -1
arrow(o, v, 'b'); hold on;
arrow(v, w, 'r'); arrow(o, u, 'm')
axis equal; grid on
axis([-1 6 -2 4])
%
echo off; diary off
```



s664x10 [664/10 revisited]

Let $\mathbf{a} = 6\mathbf{i} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$. Find $\|\mathbf{a}\|$, $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $2\mathbf{a}$, and $3\mathbf{a} + 4\mathbf{b}$.

Solution

MATLAB easily renders the needful; same with the TI-89. (NOTE: The **len** command is not built into MATLAB. I wrote it for you as part of the effort to convert the VecCalc package to YAP (Yet Another Platform)—MATLAB. Here is a list of the versions I've written over the years. (The MATLAB port will go on from week-to-week during the Spring 2004 term.)

Maple	1994
HP 48GX	1995
TI-89	1998
HP 49G	1999
HP 49g+	2004
MATLAB	2002, 2004

```
%
% Stewart 664/10
%
a = [6 0 1]; b = [1 -2 7]; % NUMERICAL vectors
length_of_a = len(a) % sqrt(37) as a decimal
length_of_a =
    6.0828
vector_sum = a + b
vector_sum =
     7    -2     8
vector_difference = a - b
vector_difference =
     5     2    -6
scalar_multiple_of_a = 2*a
scalar_multiple_of_a =
    12     0     2
linear_combination_of_a_and_b = 3*a + 4*b
linear_combination_of_a_and_b =
    22    -8    31
%
a = sym([6 0 1]) % a SYMBOLIC vector

a =

[ 6, 0, 1]

exact_length_of_a = len(a) % FORTRANesque...

exact_length_of_a =

37^(1/2)

pretty(exact_length_of_a) % ...that's nicer!

                                        1/2
                                      37
%
echo off; diary off
```

s664x14 [664/14 revisited]

Find the unit vector that has the same direction as $\mathbf{a} = 2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$.

Solution

MATLAB makes short work of this one too. Once again, the **unitvec** command is one that I wrote as part of the MATLAB VecCalc package. For help on any MATLAB command, type **"help *command*"** (without the quotes) at the MATLAB command prompt. Here *command* is the command of interest. To see the actual code (if it is available), type **"type *command*."**

```
%
% Stewart 664/14
%
v = [2 -4 7]; % NUMERICAL vector
v_hat = unitvec(v) % unit vector as a decimal
v_hat =
    0.2408   -0.4815    0.8427
%
v = sym([2 -4 7]); % SYMBOLIC vector
v_hat = unitvec(v); % exact unit vector
pretty(v_hat)

                  [        1/2              1/2            1/2]
                  [2/69 69       - 4/69 69       7/69 69     ]
%
echo off; diary off
```

s664x18 [664/18 revisited]

Given $\mathbf{a} = [-1, -2, -3]$ and $\mathbf{b} = [2, 8, -6]$, find the dot product $\mathbf{a} \cdot \mathbf{b}$.

Solution

```
%
% Stewart 664/18
%
a = [-1 -2 -3]; b = [2 8 -6];
a_dot_b = dot(a,b)
a_dot_b =
     0
%
echo off; diary off
```

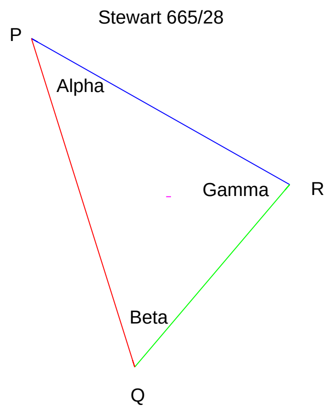
s665x28

Find, correct to the nearest degree, the three angles of the triangle with vertices $P(0, -1, 6)$, $Q(2, 1, -3)$, and $R(5, 4, 2)$.

Solution

Render the sides of the triangles as vectors, then use another VecCalc command I wrote, **angvecdg**, which gives the angle between two vectors in decimal degrees. If you look at the code, it ultimately uses a formula from the summary.

```
%
% Stewart 665/28
%
P = [0 -1 6]; Q = [2 1 -3]; R = [5 4 2];
PQ = Q-P, QR = R-Q, RP = P-R
PQ =
     2     2    -9
QR =
     3     3     5
RP =
    -5    -5     4
Alpha = angvecdg(PQ, -RP)
Alpha =
   43.0574
Beta = angvecdg(QR, -PQ)
Beta =
   57.7619
Gamma = angvecdg(RP, -QR)
Gamma =
   79.1807
sum_of_angles = Alpha + Beta + Gamma
sum_of_angles =
   180
%
echo off; diary off
```



s665x40

Find the direction cosines and direction angles (to the nearest degree) of the vector $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$.

Solution

Three commands render the needful. Radians-to-degree (**r2d**) is another VecCalc command.

```
%
% Stewart 665/40
%
v = [3 5 -4];
u = unitvec(v) % direction cosines
u =
    0.4243    0.7071   -0.5657
a = r2d(acos(u)) % direction angles in degrees
a =
   64.8959   45.0000  124.4499
%
echo off; diary off
```

s665x48 [665/48 revisited]

Find the scalar and vector projections of $\mathbf{b} = \mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$ onto $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

Solution

The **double** command converts an object to a double precision floating point decimal.

```
%
% Stewart 665/48
%
a = sym([2 -3 1]); b = sym([1 6 -2]); % SYMBOLIC vectors
c = comp(a,b); pretty(c) % exact scalar projection

                                          1/2
                                  - 9/7 14
format short % the default
c_floated = double(c) % decimal approximation
c_floated =
   -4.8107
format long
c_floated % full floating point precision
c_floated =
  -4.81070235442364
p = proj(a,b) % exact vector projection

p =

[ -18/7,  27/7,  -9/7]

format short % Restore default
% "Good enough for government work."
p_floated = double(p)
p_floated =
   -2.5714    3.8571   -1.2857
%
echo off; diary off
```

s665x59

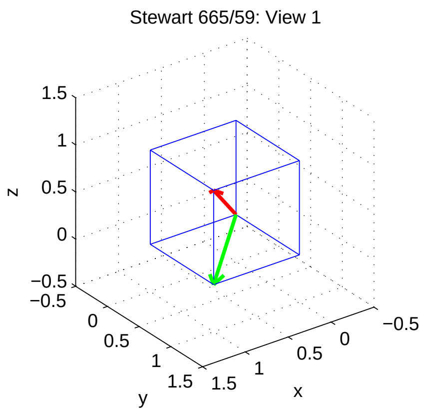
Find the angle between the diagonal of a cube and a diagonal of one of its faces.

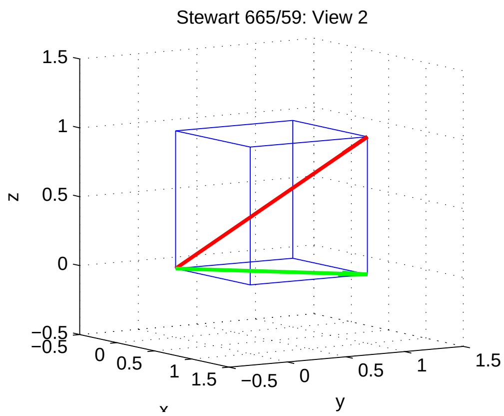
Solution

Place the cube in the first octant with one of its corners at the origin in $\mathbb{R}^3$. Let side length of the cube be $a > 0$. The endpoint of the diagonal through the cube is $A(a, a, a)$. Let $\mathbf{w} = [a, a, a]$ be the position vector from the origin to A. The origin and $B(a, a, 0)$ form a diagonal along one of the faces of the cube. Let $\mathbf{z} = [a, a, 0]$ be the position vector from the origin to B. Now simply compute the angle between $\mathbf{w}$ and $\mathbf{z}$.

```
%
% Stewart 665/59
%
syms a positive % Symbolic variable assumed to be positive.
w = [a a a]; z = [a a 0];
our_angle = double(angvecdg(w,z))
our_angle =
   35.2644
%
echo off; diary off
```

Here are two views of a cube with side length 1 together with the relevant diagonals.


Stewart 665/59: View 1
z
y
x
1.5
1
0.5
0
−0.5


Stewart 665/59: View 2
z
y
1.5
1
0.5
0
−0.5