Section 14.1
Vector Functions and Space Curves

"Functions whose range does not consists of numbers"

A bulk of elementary mathematics involves the study of functions - "rules that assign to a given input a particular output". In nearly all mathematics, these functions have had as their inputs and outputs real numbers (like $f(x) = x^2$). In this section, we introduce the idea of a vector function - a function whose outputs are vectors.

1. Vector Function Basics

We start with the formal definition of a vector function.

Definition 1.1. A vector-valued function, or a vector function, is a function whose domain is a set of real numbers and whose range is a set of vectors.

We have already seen lots of examples of vector functions.

Example 1.2. Let

$$\vec{r}(t) = (2+2t)\vec{i} + (2+2t)\vec{j} + (2-t)\vec{k}.$$

Recall that this is a vector equation for a line which passes through the point $(2, 2, 2)$ and points in the direction of $2\vec{i} + 2\vec{j} - \vec{k}$. It is also a vector function with independent variable t.

In general, a vector function in 3-space can be written in component form just like equations for lines i.e. any vector function is of the form $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ where $f(t)$, $g(t)$, and $h(t)$ are scalar functions of t. We call these function the **component functions** of the vector function $\vec{r}(t)$. As with regular functions, the usual definitions apply such as **domain**, **range** etc. We can also define other notions such as limits as continuity in of a vector function in terms of the component functions.

Definition 1.3. If $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$, then

$$\lim_{t\to a} \vec{r}(t) = \lim_{t\to a} f(t)\vec{i} + \lim_{t\to a} g(t)\vec{j} + \lim_{t\to a} h(t)\vec{k}$$

provided this limit exists.

Definition 1.4. We say $\vec{r}(t)$ is continuous at $t = a$ if

$$\lim_{t\to a} \vec{r}(t) = \vec{r}(a).$$

We illustrate with a couple of examples.

Example 1.5. Suppose that

$$\vec{r}(t) = \frac{\sin(t)}{t}\vec{i} + \sqrt{(2-t)}\vec{j} + \ln(t+1)\vec{k}.$$

Answer the following questions.

(i) What is the domain of $\vec{r}(t)$?

We must look at the domains of the component functions and take what is common to them all. The domain of the first is all real numbers but 0, the domain of the second all numbers less than or equal to 2 and the domain of the third, all real numbers greater than -1. Therefore, the domain will be $(-1, 0) \cup (0, 2]$.

(ii) Find the limit $\lim_{t\to 0} \vec{r}(t)$.

We find the limit of the component functions:

$$\lim_{t\to 0}\left(\frac{\sin(t)}{t}\vec{i} + \sqrt{(2-t)}\vec{j} + \ln(t+1)\vec{k}\right)$$

$$= \lim_{t\to 0}\frac{\sin(t)}{t}\vec{i} + \lim_{t\to 0}\sqrt{(2-t)}\vec{j} + \lim_{t\to 0}\ln(t+1)\vec{k} = \vec{i} + \sqrt{2}\vec{j}.$$

(iii) Is $\vec{r}(t)$ continuous at $t = 0$?

No - $\vec{r}(t)$ is not defined at $t = 0$, so it could not possible be continuous at $t = 0$.

2. Vector Functions and Space Curves

A space curve is a curve in space. There is a close connection between space curves and vector functions. Specifically, we can determine a vector function which traces along a space curve C (provided we put the tail of the vectors at the origin, so they are position vectors). Likewise, any vector function defines a space curve. This can be described as follows:

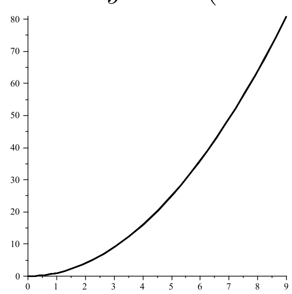
- Suppose C is a curve in space. Then we can determine parametric equations for C (equations which tell us the coordinates of a particle traveling along C at a given time t). Suppose $x = f(t)$, $y = g(t)$ and $z = h(t)$ are parametric equations for C.
- Define a vector function $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ which we shall call a vector function of C. We claim that as t varies, the position vector $\vec{r}(t)$ traces out the curve C.
- To see this, observe that any point on C has coordinates $(f(t), g(t), h(t))$, and any position vector $\vec{r}(t)$ has head at the point $(f(t), g(t), h(t))$.

- Equivalently, if $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ is a vector function, then it traces out the curve C with parametric equations $(f(t), g(t), h(t))$.

We illustrate with some examples.

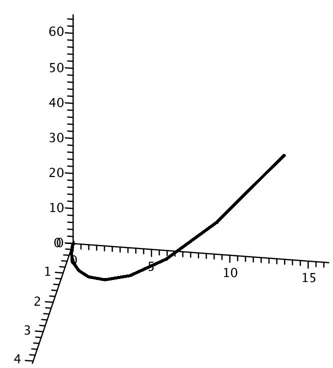
Example 2.1. Sketch the curve with vector equation $\vec{r}(t) = t^2\vec{i} + t^4\vec{j}$ in 2-space.

Observe that we have $x = t^2$, and $y = t^4 = x^2$, so this curve will be the right hand side of the parabola $y = x^2$ (WHY?).



Example 2.2. Use your last answer to sketch the curve $\vec{r}(t) = t^2\vec{i} + t^4\vec{j} + t^6\vec{k}$ in 3-space.

We know that the projection onto the xy-plane will travel along the curve $y = x^2$ in the first quadrant. Now note that $z = t^6$ just means it is always positive in the z-direction, goes to 0 at $t = 0$ and gets large quickly. A graph will look like the following:



The method we used for the last example can be very helpful when trying to draw space curves - we project down to the xy-plane (or some other plane) and draw the 2-dimensional projection and then use that to draw the space curve. We look at a couple more examples.

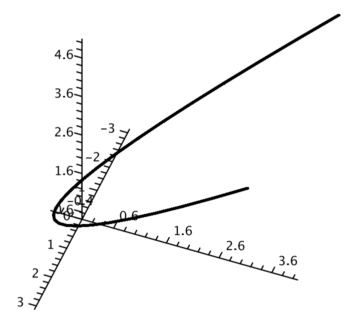
Example 2.3. Find the vector equation for the line segment between $P(1, 2, 3)$ and $Q(2, 3, 1)$.

We have already done this in an earlier section: we take

$$\vec{r}(t) = (1-t)(\vec{i} + 2\vec{j} + 3\vec{k}) + t(2\vec{i} + 3\vec{j} + \vec{k}).$$

Example 2.4. Find a vector function which represents the curve C of intersection of the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 1+y$ and then sketch this curve.

Since both equations are for z, we can substitute and eliminate: $\sqrt{x^2+y^2} = 1+y$, so $x^2+y^2 = 1+2y+y^2$ giving $y = (x^2-1)/2$. This means that all equations rely on x, so let $x = t$, so $y = (t^2-1)/2$ and $z = (t^2+1)/2$. The curve will look like the following:



Example 2.5. Suppose two particles are traveling in space, one along the curve $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ and the other along $\vec{s}(t) = (1+2t)\vec{i} + (1+6t)\vec{j} + (1+14t)\vec{k}$. Do they ever collide? Do their paths intersect?

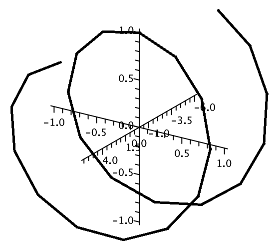
To collide, they must be at the same point at the same time i.e we must have $t = 1+2t$, $t^2 = 1+6t$ and $t^3 = 1+14t$. Solving the first equation, we have $t = -1$. However, substituting into the second equation, we get $1 = (-1)^2 = 1-6 = -5$, which is obviously not true, so these particles cannot collide.

To see if their paths intersect, we need to check if there exists s and t such that $s = 1+2t$, $s^2 = 1+6t$ and $s^3 = 1+14t$ i.e. they pass through the same point, but not necessarily at the same time. This would mean: $(1+2t)^2 = 1+6t$ and $(1+2t)^3 = 1+14t$. Solving further, we have $4t^2+4t+1 = 1+6t$, so $4t^2-2t = 0$ or $2t(t-1) = 0$ or $t = 0$ or 1. Substituting into the last equation, $t = 0$ is a solution, but not $t = 1$. When $t = 0$, we have $s = 1$, so these cross paths at $(1,1,1)$.

Example 2.6. Describe the curve defined by the vector equation

$$\vec{r}(t) = t\vec{i} + \sin(t)\vec{j} + \cos(t)\vec{k}.$$

We can use similar techniques to our previous observations, but in this case, instead of projecting down onto the xy-plane, we project down onto the yz-plane. Note that in this plane, the parametric equations $y(t)$ and $z(t)$ trace out a circle. Therefore, when we include the parametric equation for the x-coordinate, the result will be a helix extending out in the x direction (since x simply increases linearly as t increases). It will look something like the following:



Example 2.7. Describe the curve defined by the vector equation

$$\vec{r}(t) = t\vec{i} + t\sin(t)\vec{j} + t\cos(t)\vec{k}.$$

This is similar to the previous question. The effect of multiplying by t will do two things. First, as $|t|$ gets larger, the projection of the points in the yz-plane will start at $(0, 0)$ and then gradually rotate around the origin as $|t|$ gets larger moving further away from the origin as $|t|$ grows. Second, if $t > 0$, the motion will be counterclockwise, and if $t < 0$, the motion will be clockwise. In particular, if a particle were to trace out this curve, it would stop at the origin and change directions. As before, when we include the parametric equation for the x-coordinate, the result will be a helix increasing in radius extending out in the x direction (since x simply increases linearly as t increases).