EXAM 1, M312, Section 30353, 9/27/13

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Show your work. Simplify answers when possible. No books, notes, calculators are allowed. Use back sides as scratch paper (they will not be graded).

Do not write here

1 __9__ /10

2 __10__ /10

3 __10__ /10

4 __10__ /10

5 __9__ /10 raw total __48__ /50 standarized __97__ /100 letter __A+__

1. (10 pts) Find a parametrization of the curve which is the intersection of surfaces $x = z$ and $y^2 = z^3$ from the point $(1, 1, 1)$ to $(4, 8, 4)$. Find the length of this curve.

$x = z$
$y^2 = z^3$, Let $x = t$, then $z = t$ and $y = t^{3/2}$,

So a parametrization is $\boxed{c(t) = (t, t^{3/2}, t)}$ Domain of c?

the length is defined as $L(c) = \int_{t_0}^{t_1} \|c'(t)\| \, dt$.

In this case, $c'(t) = \langle 1, \frac{3}{2} t^{1/2}, 1 \rangle$.

Also, the limits of integration are given by.

$c(t_0) = \langle 1, 1, 1 \rangle = \langle t_0, t_0^{3/2}, t_0 \rangle \Rightarrow \boxed{t_0 = 1}$

$c(t_1) = \langle 4, 8, 4 \rangle = \langle t_1, t_1^{3/2}, t_1 \rangle \Rightarrow \boxed{t_1 = 4}$

therefore, the length is

$\int_1^4 \|\langle 1, \frac{3}{2} t^{1/2}, 1 \rangle\| \, dt = \int_1^4 \sqrt{\frac{9}{4} t + 2} \, dt$; Substitute: $u = \frac{9}{4} t + 2$ $du = \frac{9}{4} dt \Rightarrow dt = \frac{4}{9} du$

$\rightarrow \int \sqrt{u} \, \frac{4}{9} \, du = \frac{4}{9} \cdot \frac{2}{3} \left[u^{3/2} \right] = \frac{8}{27} u^{3/2}$ ~ substituting back

$= \frac{8}{27} \left[\left(\frac{9}{4} t + 2 \right)^{3/2} \right]_1^4 = \frac{8}{27} \left[(11)^{3/2} - \left(\frac{17}{4} \right)^{3/2} \right] = \boxed{\frac{8}{27} \left[11\sqrt{11} - \frac{17}{8} \sqrt{17} \right]}$

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2. (10 pts) Let $\mathbf{F}(x, y, z) = (e^{xy}, \sin(yz), x^2y^5z^3)$. Find the divergence and curl of $\mathbf{F}$.

$$\text{div } F = \nabla \cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle e^{xy}, \sin(yz), x^2y^5z^3 \right\rangle$$

$$= \frac{\partial}{\partial x}(e^{xy}) + \frac{\partial}{\partial y}(\sin(yz)) + \frac{\partial}{\partial z}(x^2y^5z^3)$$

$$= \boxed{ye^{xy} + z\cos(yz) + 3x^2y^5z^2}$$ ✓

$$\text{curl } F = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & \sin(yz) & x^2y^5z^3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(x^2y^5z^3) - \frac{\partial}{\partial z}(\sin(yz))\right) - \hat{j}\left(\frac{\partial}{\partial x}(x^2y^5z^3) - \frac{\partial}{\partial z}(e^{xy})\right) + \hat{k}\left(\frac{\partial}{\partial x}(\sin(yz)) - \frac{\partial}{\partial y}(e^{xy})\right)$$

$$= \hat{i}\left(5x^2y^4z^3 - y\cos(yz)\right) - \hat{j}\left(2xy^5z^3 - 0\right) + \hat{k}\left(0 - xe^{xy}\right)$$

$$= \boxed{\left\langle 5x^2y^4z^3 - y\cos(yz), -2xy^5z^3, -xe^{xy} \right\rangle}$$ ✓

3. (10 pts) Prove that $\mathbf{F}(x, y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ is a gradient vector field on $\mathbb{R}^2$ away from the origin. Find f with $\mathbf{F} = \nabla f$. Use this to compute $\int_{\mathbf{c}} \mathbf{F} \cdot ds$, where $\mathbf{c}$ is a curve connecting $(0, 1)$ with $(2, 0)$.

F(x,y) is a gradient field and

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{-2xy}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = \frac{\partial F_2}{\partial x}; \text{ for } (x,y) \neq (0,0)$$

Let us find its potential f:

$$\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2} \Rightarrow f(x,y) = \int \frac{\partial f}{\partial x}\, \partial x = \int \frac{x}{x^2+y^2}\, \partial x \text{ ; substitute } x^2+y^2 = u$$

$$\frac{\partial u}{\partial x} = 2x\, \partial x$$

$$\rightsquigarrow \int \frac{1}{u} \frac{\partial u}{2} = \frac{1}{2} \int \frac{1}{u} \partial u = \frac{1}{2}\log(u) \text{, changing back:}$$

$$\int \frac{x}{x^2+y^2}\, \partial x = \frac{1}{2}\log(x^2+y^2) + g(y) \text{, where } g \text{ is a pure function of } y.$$

But then, $\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2} = \frac{\partial}{\partial y}\left(\frac{1}{2}\log(x^2+y^2) + g(y)\right) = \frac{1}{2}\frac{2y}{x^2+y^2} + g'(y) \Rightarrow g'(y) = 0$

$\Rightarrow g(y) = C$, for C a constant.

Hence, $\boxed{f(x,y) = \frac{1}{2}\log(x^2+y^2) + C}$

to compute $\int_C F \cdot ds$, we use the fundamental theorem of gradient fields:

$\int_C F \cdot ds = f(c(b)) - f(c(a))$, where $c(a)$ and $c(b)$ are the endpoints

thus, $\int_C F \cdot ds = f(2,0) - f(1,0) = \frac{1}{2}\log(4) - \cancel{\frac{1}{2}\log(1)} = \frac{1}{2}\log(4)$

$= \frac{1}{2}\log(2^2) = \boxed{\log(2)}$

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4. (10 pts) Compute $\int_c (x^2dx + dy - xzdz)$, where c is the line segment from $(0,1,0)$ to $(1,1,1)$.

$c(t) = (1-t)(0,1,0) + t(1,1,1) = \langle t, 1, t \rangle$; $0 \leq t \leq 1$

Hence, $dx = 1$; $dy = 0$; $dz = 1$.

$\int_C x^2dx + dy - xzdz = \int_0^1 t^2 + 0 - t^2 \, dt = \int_0^1 0 = \boxed{0}$

✓

5. (10 pts) Find a parametrization of the surface $x^3 + 4xy + z = 0$ and use it to find the equation for the tangent plane at $(1, -1, 3)$.

We want to find a parametrization $\Phi(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

A possible parametrization is to set $x(u,v) = u$, $y(u,v) = v$, then $z(u,v) = -u^3 - 4uv$. In this manner the equation is satisfied:

$$x^3 + 4xy + z = (u)^3 + 4(u)(v) + (-u^3 - 4uv) = \cancel{u^3} + \cancel{4uv} - \cancel{u^3} - \cancel{4uv} = 0.$$

Hence, $\Phi(u,v) = \langle u, v, -u^3 - 4uv \rangle$.

For the tangent plane:

$T_u = \langle 1, 0, -3u^2 - 4v \rangle$; $T_v = \langle 0, 1, -4u \rangle$; so the normal vector is

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$$T_u \times T_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -3u^2 - 4v \\ 0 & 1 & -4u \end{vmatrix} = \hat{i}(3u^2 - 4v) - \hat{j}(-4u) + \hat{k}(0) = \boxed{\langle 3u^2 - 4v, 4u, 0 \rangle = \vec{n}(u,v)}$$

At the point $(1, -1, 3)$, the parameters u and v are

$$\Phi(u_0, v_0) = (1, -1, 3) = \langle u_0, v_0, -u_0^3 - 4u_0 v_0 \rangle \Rightarrow \boxed{\begin{matrix} u_0 = 1 \\ v_0 = -1 \end{matrix}}$$

So, the normal vector at this point is $\vec{n}(1,-1) = \langle 3(1)^2 - 4(-1), 4(1), 0 \rangle$

$\vec{n}(1,-1) = \langle 7, 4, 0 \rangle$.

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the tangent plane satisfies:

$$\vec{n} \cdot \langle x - 1, y + 1, z - 3 \rangle = 0$$

$$\langle 7, 4, 0 \rangle \cdot \langle x - 1, y + 1, z - 3 \rangle = 0$$

$$7x - 7 + 4y + 4 = 0$$

$$\boxed{7x + 4y - 3 = 0}$$