Extra Lecture, M312, Section 30353
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One cannot express the indefinite integral

$$\int e^{-x^2} dx$$

in terms of elementary functions, but one can prove using polar coordinates that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{1}$$

Similarly, one cannot compute indefinite integrals

$$\int \cos(x^2) dx, \quad \int \sin(x^2) dx,$$

but one can in fact determine the values of the definite integrals

$$\int_0^\infty \cos(x^2) dx, \quad \int_0^\infty \sin(x^2) dx, \tag{2}$$

For this we will use the following result:

Theorem 1. *If* $\mathbf{F} = (F_1, F_2)$ *is a* C^1 *vector field on* $\mathbb{R}^2$ *then the following are eqivalent:*

i) $\mathbf{F}$ *is a gradient field (that is* $F = \nabla f$ *for some scalar* C^2 *function* f*);*
ii) $\operatorname{curl} \mathbf{F} = 0$*;*
iii) $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$.

Proof. We can easily prove that i)⇒ii)⇔iii), so it remains to show the implication iii)⇒i). To do this we have to solve (in f) the system of equations

$$\begin{cases} \dfrac{\partial f}{\partial x} = F_1 \\[2ex] \dfrac{\partial f}{\partial y} = F_2. \end{cases} \tag{3}$$

Integrating the first equation with respect to x we see that the solution must be of the form

$$f(x,y) = \int_0^x F_1(s,y)\, dt + c(y)$$

for some function $c(y)$. Differentiating this with respect to y and using iii) we will get

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) &= \int_0^x \frac{\partial F_1}{\partial y}(s,y) + c'(y)\\ &= \int_0^x \frac{\partial F_2}{\partial x}(s,y) + c'(y)\\ &= F_2(x,y) - F_2(0,y) + c'(y),\end{aligned}$$

where the last equality follows from the fundamental theorem of calculus. We see that f satisfies the second equation in (3) if

$$c'(y) = F_2(0,y).$$

We may thus choose

$$c(y) = \int_0^y F_2(0,t)\,dt$$

and therefore

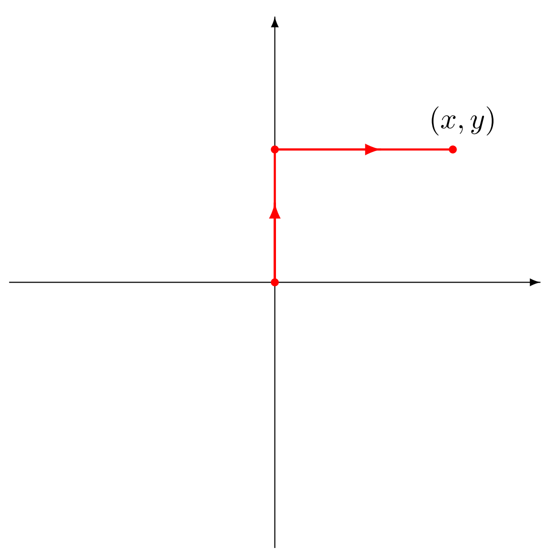
$$f(x,y) = \int_0^y F_2(0,t)\,dt + \int_0^x F_1(s,y)\,dt \tag{4}$$

solves (3). □

Remark. The left-hand side of (4) can be also written as

$$\int_{\mathbf{p}} \mathbf{F} \cdot ds,$$

where **p** is the following path



In fact, once we know that F is a gradient field, by independence of paths we have

$$f(x,y) = \int_{\mathbf{p}} \mathbf{F} \cdot ds$$

for any path $\mathbf{p}$ starting at the origin and with the endpoint at (x,y).

To compute the integrals (2) we will make use of the following functions

$$u(x,y) = e^{y^2-x^2}\cos(2xy)$$
$$v(x,y) = -e^{y^2-x^2}\sin(2xy).$$

(In fact, $u + iv = e^{-z^2}$, where $z = x + iy$, if one writes this in terms of complex numbers.) We can easily check that u and v satisfy the following Cauchy-Riemann equations:

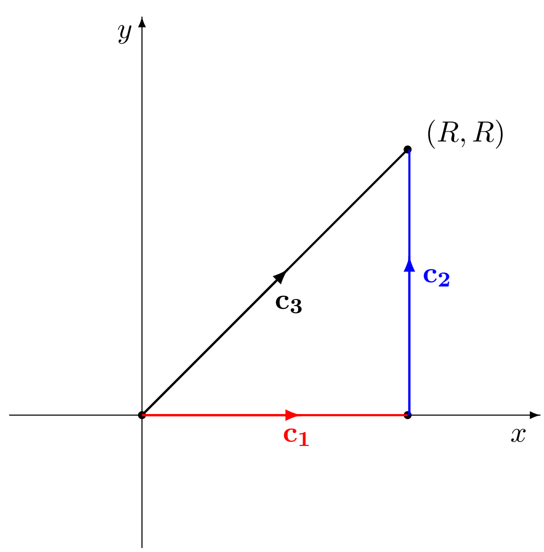
$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}. \end{cases}$$

Therefore, by Theorem 1 we get two gradient fields

$$\mathbf{F} = (v,u), \quad \mathbf{G} = (u,-v).$$

Note that we cannot find a formula for potentials of these vector fields in terms of elementary functions, we just know that they exist.

For $R > 0$ let us define the paths $\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}$ as follows



This means that we can parametrize them as follows

$$\mathbf{c}_1(t) = (t, 0), \quad 0 \le t \le R,$$
$$\mathbf{c}_2(t) = (R, t), \quad 0 \le t \le R,$$
$$\mathbf{c}_3(t) = (t, t), \quad 0 \le t \le R.$$

By independence of paths for gradient fields

$$\int_{\mathbf{c_3}} \mathbf{F} \cdot ds = \int_{\mathbf{c_1}} \mathbf{F} \cdot ds + \int_{\mathbf{c_2}} \mathbf{F} \cdot ds \tag{5}$$

and

$$\int_{\mathbf{c_3}} \mathbf{G} \cdot ds = \int_{\mathbf{c_1}} \mathbf{G} \cdot ds + \int_{\mathbf{c_2}} \mathbf{G} \cdot ds. \tag{6}$$

We can compute that

$$\int_{\mathbf{c_1}} \mathbf{F} \cdot ds = \int_{\mathbf{c_1}} (v\, dx + u\, dy) = 0, \tag{7}$$

since $v = 0$ and $dy = 0$ along $\mathbf{c_1}$. Further, since $dx = 0$,

$$\int_{\mathbf{c_2}} \mathbf{F} \cdot ds = \int_0^R e^{t^2 - R^2} \cos(2Rt)\, dt.$$

We can estimate

$$\left| \int_{\mathbf{c_2}} \mathbf{F} \cdot ds \right| \le \int_0^R e^{t^2 - R^2} dt \le \int_0^R e^{Rt - R^2} dt = \frac{1 - e^{-R^2}}{R}$$

and thus

$$\lim_{R \to \infty} \int_{\mathbf{c_2}} \mathbf{F} \cdot ds = 0. \tag{8}$$

Finally,

$$\int_{\mathbf{c_3}} \mathbf{F} \cdot ds = \int_0^R \big(-\sin(2t^2) + \cos(2t^2)\big)\, dt.$$

Combining this with (5), (7) and (8) we will get

$$\lim_{R \to \infty} \left(-\int_0^R \sin(2t^2) dt + \int_0^R \cos(2t^2) dt \right) = 0. \tag{9}$$

On the other hand, working out the similar integrals for $\mathbf{G}$ we willl obtain

$$\int_{\mathbf{c_1}} \mathbf{G} \cdot ds = \int_0^R e^{-t^2} dt,$$

$$\lim_{R \to \infty} \int_{\mathbf{c_2}} \mathbf{G} \cdot ds = 0,$$

and

$$\int_{\mathbf{c_3}} \mathbf{G} \cdot ds = \int_0^R \big(\sin(2t^2) + \cos(2t^2)\big)\, dt.$$

Combining this with (6) and (1)

$$\lim_{R\to\infty}\left(\int_0^R \sin(2t^2)dt + \int_0^R \cos(2t^2)dt\right) = \frac{\sqrt{\pi}}{2}.$$

Therefore by (9) both limits exist and

$$\int_0^\infty \cos(2t^2)dt = \int_0^\infty \sin(2t^2)dt = \frac{\sqrt{\pi}}{4}.$$

Using the substitution $x = \sqrt{2}t$ we eventually obtain

$$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dt = \frac{\sqrt{2\pi}}{8}.$$