M312 - Fall 2013 - Enrique Areyan - Review Problems Exam 2

CHAPTER 5: (p.304-305).

(15). $D=\{(x,y)\in\mathbb{R}^2 : 0\le x^2+y^2\le 1\}$. Evaluate $\iint_D f(x,y)\,dx\,dy$ in cases:

a) $f(x,y)=xy$.

$$\iint_D f(x,y)\,dx\,dy=\iint_D f(r\cos\theta,r\sin\theta)\,r\,dr\,d\theta=\int_0^{2\pi}\int_0^1 \cos\theta\sin\theta\, r^3\,dr\,d\theta=\int_0^{2\pi}\cos\theta\sin\theta\left[\frac{r^4}{4}\right]_0^1 d\theta$$

$=\frac{1}{4}\int_0^{2\pi}\cos\theta\sin\theta\,d\theta$. Integration by parts: $\int u\,dv=uv-\int v\,du$.

$u=\cos\theta$ $\quad v=\cos\theta$

$du=-\sin\theta\,d\theta$ $\quad dv=-\sin\theta\,d\theta$

$$-\int\cos\theta\sin\theta\,d\theta=\cos^2\theta+\int\cos\theta\sin\theta\,d\theta$$

$\Rightarrow -2\int\cos\theta\sin\theta\,d\theta=\cos^2\theta \therefore \int\cos\theta\sin\theta\,d\theta=-\frac{\cos^2\theta}{2}$.

We can check: $(\cos^2\theta)'=(\cos\theta\cos\theta)'=-\sin\theta\cos\theta-\sin\theta\cos\theta=-2\sin\theta\cos\theta$

$\rightsquigarrow \frac{1}{4}\int_0^{2\pi}\cos\theta\sin\theta\,d\theta=\frac{1}{4}\left[\frac{-\cos^2\theta}{2}\right]_0^{2\pi}=-\frac{1}{8}\left[\cos^2(2\pi)-\cos^2(0)\right]=\boxed{0}$

b) $f(x,y)=x^2y^2$

$$\iint_D f(x,y)\,dx\,dy=\iint_D f(r\cos\theta,r\sin\theta)\,r\,dr\,d\theta=\int_0^{2\pi}\int_0^1\cos^2\theta\sin^2\theta\, r^5\,dr\,d\theta=\frac{1}{6}\int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta$$

$$=\frac{1}{6}\left[\frac{1}{32}(4\theta-\sin(4\theta))\right]_0^{2\pi}=\frac{1}{192}\left[(8\pi-0)-(0-0)\right]=\boxed{\frac{\pi}{24}}$$

c) $f(x,y)=x^3y^3 \rightsquigarrow$ polar $f(r\cos\theta,r\sin\theta)=r^6\cos^3\theta\sin^3\theta$.

$g(\theta)=\cos^3\theta\sin^3\theta$ is an odd function over our domain:

$-g(\theta)=-\cos^3\theta\sin^3\theta=g(-\theta)=\cos^3(-\theta)\sin^3(-\theta)=\cos^3(\theta)\sin^3(-\theta)$.

$\Rightarrow$ The integral is zero.

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25) $f(x,y)=x-y$; $D=\{$triangle with vertices $(0,0),(1,0)$, and $(2,1)\}$.

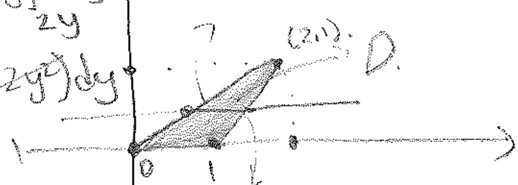
$$\int_0^1\int_{2y}^{y+1}x-y\,dx\,dy=\int_0^1\left[\frac{x^2}{2}-xy\right]_{2y}^{y+1}dy$$

$$=\int_0^1\left(\frac{y^2+2y+1}{2}-(y^2+y)\right)-\cancel{(2y^2-2y^2)}\,dy$$

$$\left[\frac{1}{2}\left(\frac{y^3}{3}+y^2+y\right)-\left(\frac{y^3}{3}+\frac{y^2}{2}\right)\right]_0^1$$

$$=\frac{1}{2}\left(\frac{1}{3}+2\right)-\left(\frac{1}{3}+\frac{1}{2}\right)=\frac{1}{6}+1-\frac{1}{3}+\frac{1}{2}=\frac{1+6-2+3}{6}=\frac{8}{6}=\boxed{\frac{4}{3}}$$

$L_1 \Rightarrow y=\frac{x}{2}$



$L_2 \Rightarrow y=x-1$

L_1: $(0,0)$, $(2,1)$.

$y=mx+b$

$0=m\cdot 0+b \Rightarrow \boxed{b=0}$

$1=2m+b \Rightarrow \boxed{m=\frac{1}{2}}$

$L_1:=\boxed{y=\frac{x}{2}}$

L_2: $(1,0)$, $(2,1)$ $\quad \boxed{y=x-1}$

$y=mx+b$

$0=m+b$ $\Rightarrow \boxed{m=1}$

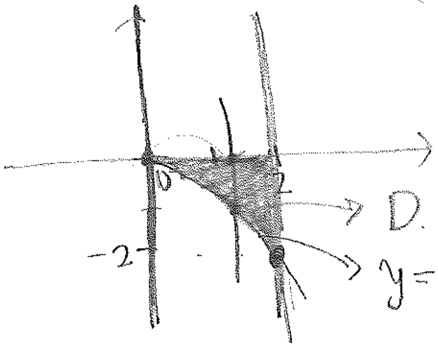
$1=m2+b$ $\quad \boxed{b=-1}$

$1=m$

7) $f(x,y) = x^2 + 2xy^2 + 2$,

$y(\frac{1}{2}) = -\frac{1}{4} + \frac{1}{2} = \frac{1}{2}$

$\iint f(x,y)\,dy\,dx =$



$y(0) = 0$
$y(1) = 0$
$y(2) = -2$

$\int_0^2 \int_{-x^2+x}^0 x^2 + 2xy^2 + 2\,dy\,dx$

$\int_0^2 \left(yx^2 + \frac{2}{3}xy^3 + 2y\right)_{-x^2+x}^{0} dx = \int_0^2 -\left[(-x^2+x)x^2 + \frac{2}{3}x(-x^2+x)^3 + 2(-x^2+x)\right]dx$

$\int_0^2 -\left[-x^4 + x^3 - \frac{2}{3}x(x^4 - 2x^3 + x^2)(-x^2+x) - 2x^2 + 2x\right]dx$

$\int_0^2 -\left[-x^4 + x^3 - \frac{2}{3}\left[-x^7 + x^6 + 2x^6 - 2x^5 - x^5 + x^4\right] - 2x^2 + 2x\right] dx$

$\int_0^2 x^4 - x^3 - \frac{2}{3}x^7 + \frac{2}{3}x^6 + \frac{4}{3}x^6 - \frac{4}{3}x^5 - \frac{2}{3}x^5 + \frac{2}{3}x^4 - 2x^2 + 2x\,dx$

$\int_0^2 \frac{5}{3}x^4 - x^3 - \frac{2}{3}x^7 + 2x^6 - 2x^5 - 2x^2 + 2x\,dx = \left[\frac{x^5}{3} - \frac{x^4}{4} - \frac{x^8}{12} + \frac{2}{7}x^7 - \frac{x^6}{3} - \frac{2}{3}x^3 + x^2\right]_0^2$

$\frac{32}{3} - \frac{16}{4} - \frac{256}{12} + \frac{256}{7} - \frac{64}{3} - \frac{16}{3} + 4 = \frac{32-64-16}{3} \quad \frac{48+256}{12} + \frac{256+28}{7}$

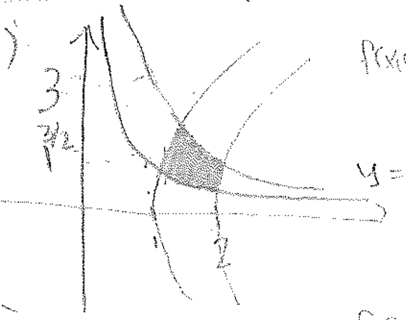
$16 - \frac{304}{12} + \frac{284}{7}$

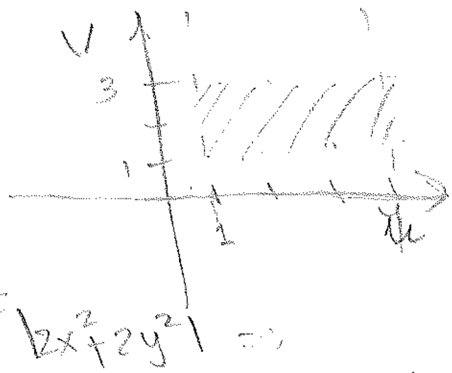
CHAPTER 6 (p. 347–348):

$u = x^2 - y^2 \qquad \frac{\partial(x,y)}{\partial(u,v)} = |$

$v = xy$

$f(x,y) = (x^2 + y^2)$





$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = |2x^2 + 2y^2| \Rightarrow$

$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2x^2+2y^2}$

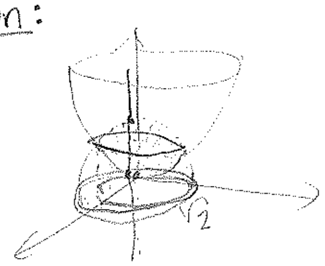
$xy = 1$
$xy = 3$
$x^2 - y^2 = 1$
$x^2 - y^2 = 4$

$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

$\int_1^3 \int_1^4 \frac{u^2+v^2}{2(u^2+v^2)}\,du\,dv = \frac{3 \cdot \cancel{2}}{\cancel{2}} = \boxed{3}$

(5) Find the volume inside the surfaces $x^2+y^2=z$ and $x^2+y^2+z^2=2$

Solution:



$$\left.\begin{array}{l} z = x^2+y^2 \quad (\text{low}) \\ z = \sqrt{2-x^2-y^2} \quad \text{high.} \end{array}\right\}$$

$$z + z^2 = 2 \Rightarrow 2z^2 = 2 \Rightarrow z^2 = 1 \Rightarrow z = \pm 1 \Rightarrow \boxed{z=1}$$

$$\iint_D \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dz\,dx\,dy = \iint_D \sqrt{2-(x^2+y^2)} - (x^2+y^2)\,dx\,dy$$

$$\int_0^{2\pi}\int_0^{\sqrt{2}} \left(\sqrt{2-r^2} - r^2\right) r\,dr\,d\theta = 2\pi \int_0^{\sqrt{2}} r\sqrt{2-r^2} - r^3\,dr.$$

$$= 2\pi \left\{ \underbrace{\int_0^1 r\sqrt{2-r^2}\,dr}_{A} - \underbrace{\int_0^1 r^3\,dr}_{B} \right\}$$

(A) $= \int_0^1 r\sqrt{2-r^2}\,dr$; $u = 2-r^2 \Rightarrow du = -2r\,dr \Rightarrow r\,dr = -\dfrac{du}{2}$

$$-\frac{1}{2}\int \sqrt{u}\,du = -\frac{1}{2}\left[\frac{2}{3}u^{3/2}\right] = -\frac{1}{3}\left(u^{3/2}\right) \rightsquigarrow -\frac{1}{3}\left[(2-r^2)^{3/2}\right]_0^1$$

$$= -\frac{1}{3}\left\{1 - 2\sqrt{2}\right\}$$

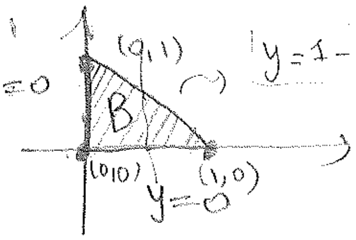
(B) $\int_0^1 r^3\,dr = \left[\dfrac{r^4}{4}\right]_0^1 = \dfrac{1}{4}$

$$= 2\pi\left\{\frac{2}{3}\sqrt{2} - \frac{1}{3} - \frac{1}{4}\right\} = 2\pi\left\{\frac{2}{3}\sqrt{2} - \frac{7}{12}\right\} = \boxed{\frac{1}{3}\pi\left\{4\sqrt{2} - \frac{7}{2}\right\}}$$

$\iint_B e^{\frac{y-x}{y+x}}\,dx\,dy$: $B = \{$ interior of triangle with vertices $(0,0), (0,1)$ and $(1,0)\}$.

$x=0,\ y=1 \Rightarrow u=1$

$x=1,\ y=0 \Rightarrow u=1$



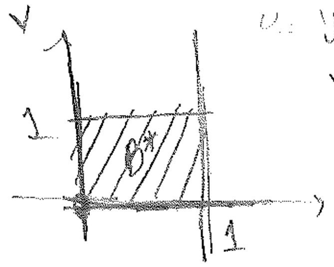
$y = 1-x \Rightarrow y+x=1$

$B = \{(x,y) \in \mathbb{R}^2 :\ 0 \le y \le 1-x;\ 0 \le x \le 1\}$

$u = x+y$

$v = x$

$0 \le y \le 1-x$



$\iint_B f(x,y)\,dx\,dy = \iint_{B^*} f(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$

$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} \Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$\begin{cases} u = x+y \\ v = y-x \end{cases}$

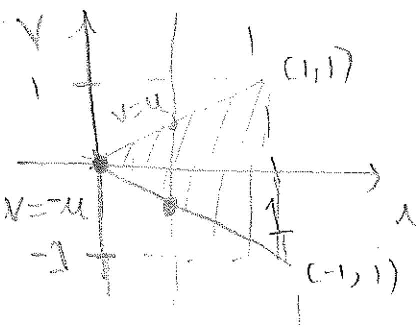
$y+x=1 \Rightarrow x = 1-y.$

$v = y-x = y-(1-y) = 2y-1$

$y=0 \Rightarrow 1-x$

$2(1-x)-1$

$1-2x$



$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|}$

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 \Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}.$

$\iint_B f(x,y)\,dx\,dy = \iint_{B^*} f(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

$= \frac{1}{2}\int_0^1\int_{-u}^{u} e^{\frac{v}{u}}\,dv\,du = \frac{1}{2}\int_0^1 \left[u\,e^{\frac{v}{u}}\right]_{-u}^{u} du$

$\frac{1}{2}\int_0^1 \{u(e^1 - e^{-1})\} = \frac{e^1-e^{-1}}{2}\left[\frac{u^2}{2}\right]_0^1 = \frac{e-\frac{1}{e}}{4} = \boxed{\frac{1}{4}\left(e-\frac{1}{e}\right)}$

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(21) Find the center of mass of the solid hemisphere $\delta(x,y,z)=$

$$V=\{(x,y,z) \mid x^2+y^2+z^2 \leq a^2 \text{ and } z \geq 0\}$$

Solution: Center of mass $(\bar{x},\bar{y},\bar{z})$ is given by: $d = \frac{m}{V}$

$m = d \cdot V$

$$\bar{x} = \frac{\iiint x\,\delta(x,y,z)\,dx\,dy\,dz}{\text{mass}(V) = \iiint \delta(x,y,z)\,dx\,dy\,dz}$$

$\frac{4}{3}\pi r^3 = \text{vol}$

$$\boxed{\begin{matrix} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{matrix}}$$

$$\text{mass}(V) = \frac{c\,\frac{4}{3}\pi a^3}{2} = \boxed{\frac{4}{6} c a^3 \pi}$$

$$\iiint_V x\,c\,dV = c\iiint_V x\,dV = c\int_0^{\pi}\int_0^{2\pi}\int_0^{a} (r\sin\theta\cos\varphi)(r^2\sin\theta)\,dr\,d\theta\,d\varphi$$

$$= c\int_0^{\pi}\int_0^{2\pi}\int_0^{a} r^3\sin^2\theta\cos\varphi\,dr\,d\theta\,d\varphi = \frac{ca^4}{4}\int_0^{\pi}\int_0^{2\pi} \sin^2\theta\cos\varphi\,d\theta\,d\varphi = \boxed{0}$$

CHAPTER 7: (p. 424-425).

(9) Write a formula for the surface area of $\Phi:(r,\theta)\mapsto(x,y,z)$, where $x=r\cos\theta$, $y=2r\sin\theta$, $z=r$; $0\leq r\leq 1$, $0\leq\theta\leq 2\pi$.

$2r\cos^2\theta + 2r\sin^2\theta$

$$\iint_D \|T_r\times T_\theta\|\,dr\,d\theta$$

$$\Phi' = \begin{pmatrix}\Phi_r \\ \Phi_\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & 2\sin\theta & 1 \\ -r\sin\theta & 2r\cos\theta & 0\end{pmatrix} \; ; \; \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = 2r \; ; \; \left|\frac{\partial(x,z)}{\partial(r,\theta)}\right| = r\sin\theta$$

$\left|\frac{\partial(y,z)}{\partial(r,\theta)}\right| = -2r\cos\theta$ the surface area is:

$$\iint_D \sqrt{4r^2 + r^2\sin^2\theta + 4r^2\cos^2\theta}\,dr\,d\theta = \int_0^{2\pi}\int_0^{1}\sqrt{r^2(4+\sin^2\theta+4\cos^2\theta)}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1} r\sqrt{4+\sin^2\theta+4\cos^2\theta}\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\sqrt{4+\sin^2\theta+4\cos^2\theta}\,d\theta$$

$$2\sin^2\theta = 1-\cos 2\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

1) Compute the integral of $f(x,y,z) = x^2 + y^2 + z^2$ over:

$\Phi(u,v) \mapsto (x,y,z)$, where:

$x = h(u,v) = u+v$, $y = g(u,v) = u$, $z = f(u,v) = v$

$0 \le u \le 1$, $0 \le v \le 1$. Sketch

$$\iint_D f(x,y,z)\, \|T_u \times T_v\|\, du\, dv.$$

$$\Phi' = \begin{pmatrix} \Phi_u \\ \Phi_v \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = -1; \quad \left|\frac{\partial(x,z)}{\partial(u,v)}\right| = 1;$$

$$\left|\frac{\partial(y,z)}{\partial(u,v)}\right| = 1.$$

$$\|T_u \times T_v\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\iint_D f(x,y,z)\|T_u \times T_v\|\, du\, dv = \iint_D f(\Phi(u,v))\sqrt{3}\, du\, dv$$

$$= \int_0^1\int_0^1 \left[(u+v)^2 + u^2 + v^2\right]\sqrt{3}\, du\, dv = \sqrt{3}\int_0^1\int_0^1 2u^2 + 2uv + 2v^2\, du\, dv$$

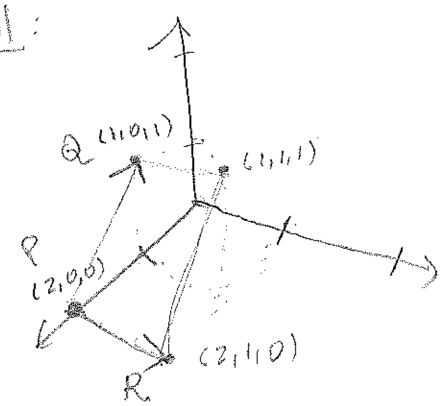
$$\sqrt{3}\int_0^1 \left[\frac{2}{3}u^3 + u^2 v + 2uv^2\right]_{u=0}^{u=1} dv = \sqrt{3}\int_0^1 \frac{2}{3} + v + 2v^2\, dv$$

$$= \sqrt{3}\left[\frac{2}{3}v + \frac{v^2}{2} + \frac{2}{3}v^3\right]_{v=0}^{v=1} = \sqrt{3}\left[\frac{2}{3} + \frac{1}{2} + \frac{2}{3}\right] = \sqrt{3}\left[\frac{4}{3} + \frac{1}{2}\right]$$

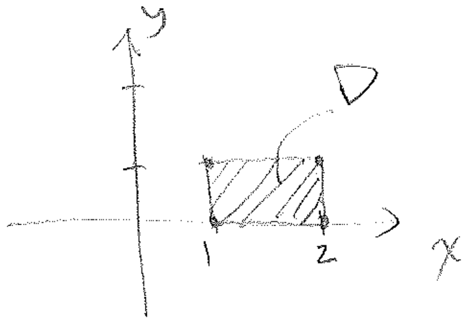
$$= \sqrt{3}\left[\frac{8+3}{6}\right] = \boxed{\frac{11}{6}\sqrt{3}}$$

(13) Compute the integral of $f(x,y,z) = xyz$ over the rectangle with vertices $(1,0,1), (2,0,0), (1,1,1)$ and $(2,1,0)$.

Sol:



Domain D:



$P = (2,0,0)$, $Q = (1,0,1)$, $R = (2,1,0)$

$PQ = (1,0,1) - (2,0,0) = (-1,0,1)$

$PR = (2,1,0) - (2,0,0) = (0,1,0)$

$$\vec{n} = PQ \times PR = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \hat{i}(-1) - \hat{j}(0) + \hat{k}(-1) = \langle -1, 0, -1 \rangle$$

Plane is: $\vec{n} \cdot (\langle x - x_0, y - y_0, z - z_0 \rangle) = 0$

$\langle -1,0,-1 \rangle \cdot \langle x-2, y, z \rangle = 0$

$2 - x - z = 0 \Rightarrow \boxed{x + z = 2}$

$\Rightarrow \boxed{z = 2 - x}$. A parametrization is: $\Phi(u,v) = (u, v, 2-u)$
$1 \le u \le 2$; $0 \le v \le 1$

$$\iint_D f(\Phi(u,v)) \, \|T_u \times T_v\| \, du\,dv.$$

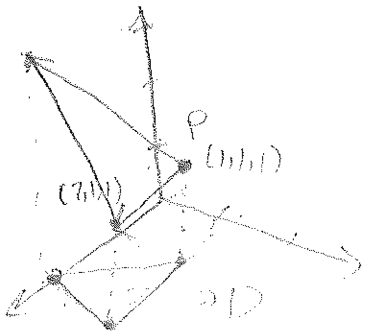
$$\Phi' = \begin{pmatrix} \Phi_u \\ \Phi_v \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1 ; \left|\frac{\partial(x,z)}{\partial(u,v)}\right| = 0 ; \left|\frac{\partial(y,z)}{\partial(u,v)}\right| = 1$$

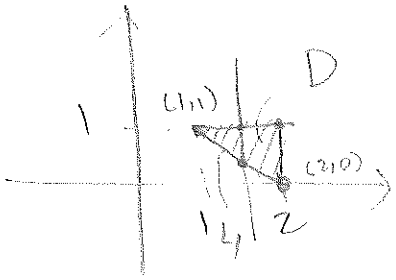
$$\int_0^1 \int_1^2 uv(2-u)\sqrt{2}\, du\,dv = \sqrt{2} \int_0^1 \int_1^2 2uv - u^2 v \, du\,dv = \sqrt{2} \int_0^1 \left[u^2 v - \frac{u^3 v}{3} \right]_1^2 dv$$

$$= \sqrt{2} \int_0^1 \left(4v - \frac{8}{3}v\right) - \left(v - \frac{v}{3}\right) dv = \sqrt{2} \int_0^1 3v - \frac{7}{3}v \, dv = \frac{2\sqrt{2}}{3} \int_0^1 v\,dv$$

$$= \frac{\cancel{2}\sqrt{2}}{3} \left[\frac{v^2}{\cancel{2}}\right]_0^1 = \boxed{\frac{\sqrt{2}}{3}}$$

5) Compute the surface integral of x over the triangle with vertices $(1,1,1)$, $(2,1,1)$, $(2,0,3)$.





L_1: $y = mx + b$

$(1,1)$: $1 = m + b$

$(2,0)$: $0 = 2m + b$

$1 = -m \Rightarrow \boxed{m = -1}$

$0 = -2 + b \Rightarrow b = 2$

$L_1 = \boxed{y = 2 - x}$

$D = \{(x,y) \in \mathbb{R}^2 : 2-x \le y \le 1,\ 1 \le x \le 2\}$.

the plane containing the triangle is:

$P = (1,1,1)$, $Q = (2,1,1)$, $R = (2,0,3)$

$PQ = (2,1,1) - (1,1,1) = (1,0,0)$

$PR = (2,0,3) - (1,1,1) = (1,-1,2)$

$$PQ \times PR = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 1 & -1 & 2 \end{vmatrix} = \hat{i}(0) - \hat{j}(2) + \hat{k}(-1) = \langle 0, -2, -1 \rangle$$

$P := \vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$\langle 0,-2,-1 \rangle \cdot \langle x-1, y-1, z-1 \rangle = 0 \Leftrightarrow \boxed{2y + z = 3}$

$2 - 2y + 1 - z = 0$. We can parametrize it.

As a graph: $z = 3 - 2y$. $1 \le u \le 2$, $2 - u \le v \le 1$

$\Phi(u,v) = \langle u, v, 3 - 2v \rangle$.

$\iint_D f(\Phi(u,v)) \|T_u \times T_v\| \, du\, dv$, where:

$\Phi' = \begin{pmatrix} \Phi_u \\ \Phi_v \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix} \to \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1;\ \left|\frac{\partial(x,z)}{\partial(u,v)}\right| = -2,\ \left|\frac{\partial(y,z)}{\partial(u,v)}\right| = 0.$

$\|T_u \times T_v\| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$$\iint_D f(\Phi(u,v)) \|T_u \times T_v\| \, du\, dv = \int_1^2 \int_{2-u}^1 u\sqrt{5}\, dv\, du = \sqrt{5} \int_1^2 u(1 - 2 + u)\, du$$

$$\sqrt{5} \int_1^2 -u + u^2\, du = \sqrt{5} \left[-\frac{u^2}{2} + \frac{u^3}{3} \right]_1^2 = \sqrt{5}\left[\left(-2 + \frac{8}{3}\right) - \left(-\frac{1}{2} + \frac{1}{3}\right) \right]$$

$$\sqrt{5}\left[-2 + \frac{1}{2} - \frac{1}{3} + \frac{8}{3} \right] = \sqrt{5}\left[-\frac{3}{2} + \frac{7}{3} \right] = \sqrt{5}\left[\frac{-9 + 14}{6} \right] = \boxed{\frac{5\sqrt{5}}{6}}$$