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1) Exercise 8.3.2.

(a) $F(x,y) = \langle \cos(xy) - xy\sin(xy), -x^2\sin(xy) \rangle$.

Here $P(x,y) = \cos(xy) - xy\sin(xy)$ and $Q(x,y) = -x^2\sin(xy)$.

$$\frac{\partial P}{\partial y} = -x\sin(xy) - [x\sin(xy) + x^2y\cos(xy)] = -2x\sin(xy) - x^2y\cos(xy).$$

$\frac{\partial Q}{\partial x} = -2x\sin(xy) - x^2y\cos(xy)$. Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow F$ is a gradient field.

Let us find its potential f.

$$\frac{\partial f}{\partial x} = \cos(xy) - xy\sin(xy) \Rightarrow f(x,y) = \int \frac{\partial f}{\partial x}\,\partial x = \int [\cos(xy) - xy\sin(xy)]\,\partial x$$

$= \frac{\sin(xy)}{y} + x\cos(xy) - \frac{\sin(xy)}{y} + g(y)$, where g is a pure function of y. $y \neq 0$.

Hence, $f(x,y) = x\cos(xy) + g(y)$.

$$\frac{\partial f}{\partial y} = -x^2\sin(xy) = \frac{\partial}{\partial y}[x\cos(xy) + g(y)] = -x^2\sin(xy) + g'(y).$$

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = c$, where c is a constant.

thus, $\boxed{f(x,y) = x\cos(xy) + c}$ ✓

(b) $F(x,y) = \langle x\sqrt{x^2y^2+1}, y\sqrt{x^2y^2+1} \rangle$

Here, $P(x,y) = x\sqrt{x^2y^2+1}$ and $Q(x,y) = y\sqrt{x^2y^2+1}$

$$\frac{\partial P}{\partial y} = \frac{x^3y}{\sqrt{x^2y^2+1}} \neq \frac{xy^3}{\sqrt{x^2y^2+1}} = \frac{\partial Q}{\partial x} \Rightarrow F \text{ is } \underline{\text{NOT}} \text{ a gradient field.}$$

✓

(c) $F(x,y) = \langle 2x\cos y + \cos y, -x^2\sin y - x\sin y \rangle$

Here, $P(x,y) = 2x\cos y + \cos y$ and $Q(x,y) = -x^2\sin y - x\sin y$.

$\frac{\partial P}{\partial y} = -2x\sin y - \sin y = \frac{\partial Q}{\partial x} \Rightarrow F$ is a gradient field.

Let us find its potential f.

$\frac{\partial f}{\partial x} = (2x+1)\cos y \Rightarrow f(x,y) = \int \frac{\partial f}{\partial x} \partial x = \int [(2x+1)\cos y]\, \partial x = \cos(y)(x^2+x) + g(y)$.

thus, $f(x,y) = \cos(y)(x^2+x) + g(y)$.

$\frac{\partial f}{\partial y} = (-x^2-x)\sin y = \frac{\partial}{\partial y}\left[\cos y\,(x^2+x) + g(y)\right] = -\sin y\,(x^2+x) + g'(y)$. thus,

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = c$, where c is a constant.

$$\boxed{f(x,y) = \cos y\,(x^2+x) + c}$$

2) Exercise 8.3.13.

Let $F(x,y,z) = \langle e^x\sin y, e^x\cos y, z^2 \rangle$ Evaluate the integral $\int_C F\cdot d\vec{s}$, where $\vec{c}(t) = (\sqrt{t}, t^3, e^{\sqrt{t}})$, $0 \le t \le 1$.

Solution: Note that this is a gradient field because:

$$\operatorname{curl} F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x\sin y & e^x\cos y & z^2 \end{vmatrix} = \vec{i}\left(\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(e^x\cos y)\right) - \vec{j}\left(\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(e^x\sin y)\right) + \vec{k}\left(\frac{\partial}{\partial x}(e^x\cos y) - \frac{\partial}{\partial y}(e^x\sin y)\right) = \langle 0,0,0 \rangle.$$

Let us find the potential f of F.

$\frac{\partial f}{\partial x} = e^x\sin y \Rightarrow f(x,y,z) = \int \frac{\partial f}{\partial x} \partial x = \int (e^x\sin y)\, \partial x = e^x\sin y + g(y,z)$.

$\frac{\partial f}{\partial y} = e^x\cos y = \frac{\partial}{\partial y}\left[e^x\sin y + g(y,z)\right] = e^x\cos y + \frac{\partial g}{\partial y} \Rightarrow \frac{\partial g}{\partial y} = 0$

$g(y,z) = \int \frac{\partial g}{\partial y} \partial y = \int 0\, \partial y = c + h(z)$, c a constant. therefore,

$$f(x,y,z) = e^x\sin y + h(z) + c.$$

$\frac{\partial f}{\partial z} = z^2 = \frac{\partial}{\partial z}\left[e^x\sin y + h(z) + c\right] = h'(z) \Rightarrow h(z) = \int z^2\, dz = \frac{z^3}{3} + d$,

d a constant. therefore

$$\boxed{f(x,y,z) = e^x\sin y + \frac{z^3}{3} + a}$$, where a is a constant

Now, the integral we want to compute can be easily calculated $\longrightarrow$

$$\int_C F\cdot d\vec{s} = f(c(1)) - f(c(0)) = f(\langle 1,1,e\rangle) - f(\langle 0,0,1\rangle)$$
$$= [e^1\sin(1) + \frac{e^3}{3}] - [e^0\cancel{\sin(0)} + \frac{1}{3}] = \boxed{e\cdot\sin(1) + \frac{e^3}{3} - \frac{1}{3}}$$

3) Exercise 8.3.18

(a) $F(x,y) = (2x+y^2-y\sin x,\ 2xyz + \cos x)$

$\frac{\partial P}{\partial y} = 2y - \sin x \neq 2yz - \sin x = \frac{\partial Q}{\partial x} \Rightarrow F$ is not a gradient field.

(b) $F(x,y,z) = (6x^2z^2, 5x^2y^2, 4y^2z^2)$.

$$\text{curl } F = \nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6x^2z^2 & 5x^2y^2 & 4y^2z^2 \end{vmatrix} = \hat{i}(8yz^2-0) - \hat{j}(0-12x^2z) + \hat{k}(10xy^2-0)$$
$$= \langle 8yz^2, 12x^2z, 10xy^2\rangle \neq \vec{0}$$

$\Rightarrow F$ is not a gradient field.

(c) $F(x,y) = (y^3+1, 3xy^2+1)$

$\frac{\partial P}{\partial y} = 3y^2 = \frac{\partial Q}{\partial x} \Rightarrow F$ is a gradient field. Let us find its potential:

$\frac{\partial f}{\partial x} = y^3+1 \Rightarrow f(x,y) = \int \frac{\partial f}{\partial x}\partial x = \int (y^3+1)\partial x = xy^3 + x + g(y)$.

$$f(x,y) = xy^3 + x + g(y).$$

$\frac{\partial f}{\partial y} = 3xy^2+1 = \frac{\partial}{\partial y}[xy^3+x+g(y)] = 3xy^2 + g'(y) \Rightarrow g'(y) = 1 \Rightarrow g(y) = y + c$, c a constant

$$\boxed{f(x,y) = xy^3 + x + y + c}$$

(d) $F(x,y,z) = (xe^{(x^2+y^2)} + 2xy,\ ye^{(x^2+y^2)} + 4y^3z,\ y^4)$.

$$\text{curl } F = \nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^{(x^2+y^2)}+2xy & ye^{(x^2+y^2)}+4y^3z & y^4 \end{vmatrix}$$

$$= \hat{i}(4y^3-4y^3) - \hat{j}(0-0) + \hat{k}\left(2xye^{(x^2+y^2)} - \left(2xye^{(x^2+y^2)} + 2x\right)\right)$$

$= \langle 0,0,-2x\rangle \neq \vec{0} \Rightarrow F$ is not a gradient field.

(4) Exercise 8.3.19.

Show that the following vector fields are conservative.
Calculate $\int_C F \cdot d\vec{s}$ for the given curve.

(a) $F = \langle xy^2 + 3x^2y,\ (x+y)x^2 \rangle$.

$\frac{\partial P}{\partial y} = 2xy + 3x^2 = \frac{\partial Q}{\partial x} \Rightarrow$ F is a gradient field $\Rightarrow$ F is conservative.

Let us find the potential:

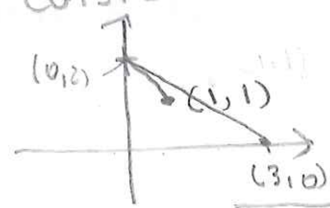
$\frac{\partial f}{\partial x} = xy^2 + 3x^2y \Rightarrow f(x,y) = \int \frac{\partial f}{\partial x}\, \partial x = \int (xy^2 + 3x^2y)\, \partial x = \frac{x^2y^2}{2} + x^3y + g(y)$.

Hence: $f(x,y) = \frac{x^2y^2}{2} + x^3y + g(y)$.

$\frac{\partial f}{\partial y} = (x+y)x^2 = \frac{\partial}{\partial y}\left[\frac{x^2y^2}{2} + x^3y + g(y)\right] = x^2y + x^3 + g'(y)$ then,

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = c$, where c is a constant.

$$\boxed{f(x,y) = \frac{x^2y^2}{2} + x^3y + c}$$



Let us evaluate $\int_C F \cdot d\vec{s}$, where C is the curve

$\int_C F\, d\vec{s} = f(3,0) - f(1,1) = \left(\frac{3^2 0^2}{2} + 3^3 \cdot 0 + c\right) - \left(\frac{1^2 1^2}{2} + 1^3 1 + c\right) = -\frac{1}{2} - 1 = \boxed{-\frac{3}{2}}$

(b) $F = \left\langle \frac{2x}{y^2+1},\ \frac{-2y(x^2+1)}{(y^2+1)^2} \right\rangle$

$\frac{\partial P}{\partial y} = \frac{-4xy}{(y^2+1)^2} = \frac{\partial Q}{\partial x} \Rightarrow$ F is a gradient field $\Rightarrow$ F is conservative

Let us find the potential:

$\frac{\partial f}{\partial x} = \frac{2x}{y^2+1} \Rightarrow f(x,y) = \int \frac{\partial f}{\partial x} \cdot \partial x = \int \left(\frac{2x}{y^2+1}\right) \partial x = \frac{x^2}{y^2+1} + g(y)$.

Hence: $f(x,y) = \frac{x^2}{y^2+1} + g(y)$.

?

$\frac{\partial f}{\partial y} = \frac{-2y(x^2+1)}{(y^2+1)^2} = \frac{\partial}{\partial y}\left[\frac{x^2}{y^2+1} + g(y)\right] = \frac{-2y(x^2+1)}{(y^2+1)^2} + g'(y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = C$, for C a constant.

$$\boxed{f(x,y) = \frac{x^2}{y^2+1} + C}$$

Let us evaluate $\int_C F \cdot d\vec{s}$, where $c(t) = \langle t^3 - 1, t^6 - t \rangle$, $0 \le t \le 1$

$\int_C F\, d\vec{s} = f(c(1)) - f(c(0)) = f((0,0)) - f((-1,0)) = \frac{0^2}{0^2+1} - \frac{(-1)^2}{0^2+1} = \boxed{-1}$

-1

(c) $F = \langle \cos(xy^2) - xy^2\sin(xy^2), -2x^2y\sin(xy^2)\rangle$

$\frac{\partial P}{\partial y} = -2xy\sin(xy^2) - [2xy\sin(xy^2) + 2x^2y^2\cos(xy^2)] = -4xy\sin(xy^2) - 2x^2y^2\cos(xy^2) = \frac{\partial Q}{\partial x}$

Let us evaluate $\int_C F\cdot d\vec{s}$, where $c(t) = (e^t, e^{t+1})$, $-1 \le t \le 0$.

By first finding the potential:

$\frac{\partial f}{\partial x} = \cos(xy^2) - xy^2\sin(xy^2) \Rightarrow f(x,y) = \int \frac{\partial f}{\partial x}\,\partial x = \int [\cos(xy^2) - xy^2\sin(xy^2)]\,\partial x$

$= \cancel{\frac{\sin(xy^2)}{y^2}} + x\cos(xy^2) - \cancel{\frac{\sin(xy^2)}{y^2}} + g(y)$. Hence,

$f(x,y) = x\cos(xy^2) + g(y)$

$\frac{\partial f}{\partial y} = -2x^2y\sin(xy^2) = \frac{\partial}{\partial y}[x\cos(xy^2) + g(y)] = -2x^2y\sin(xy^2) + g'(y)$.

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C$, where C is a constant.

$\boxed{f(x,y) = x\cos(xy^2) + C}$ then,

$\int_C F\cdot d\vec{s} = f(c(0)) - f(c(-1)) = f((e^0, e^1)) - f((e^{-1}, e^0)) = f((1,e)) - f((e^{-1}, 1))$

$= [1\cos(1e^2) + C] - [e^{-1}\cos(e^{-1}1^2) + C] = \boxed{\cos(e^2) - \frac{\cos(\frac{1}{e})}{e}}$

5.) Exercise 8.4.2. Verify the divergence theorem:

$W = [0,1]\times[0,1]\times[0,1]$, $F = \langle zy, xz, xy\rangle$.

We want to verify:

$$\iiint_W (\nabla\cdot F)\,dV = \iint_{\partial W} F\cdot d\vec{S}.$$

LHS: $\iiint_W \left[\frac{\partial}{\partial x}(zy) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy)\right]dV = \iiint_W 0\,dV = \boxed{0}$

RHS: $\iint_{\partial W} F\cdot d\vec{S} = -\iint_{S_1} xy\,dS + \iint_{S_2} xy\,dS - \iint_{S_3} zy\,dS + \iint_{S_4} zy\,dS - \iint_{S_5} xz\,dS + \iint_{S_6} xz\,dS$

↑ calculations on ex. 2 page 462-463

$S_1 : z=0$, $0 \le x \le 1$, $0 \le y \le 1$. S_1: $\Phi(u,v) = (u,v,0)$; $0 \le u \le 1$, $0 \le v \le 1$
$S_2 : z=1$, $0 \le x \le 1$, $0 \le y \le 1$. S_2: $\Phi(u,v) = (u,v,1)$: $0 \le u \le 1$, $0 \le v \le 1$.

$$-\iint_{S_1} xy\, dS + \iint_{S_2} xy\, dS = -\int_0^1\int_0^1 uv\, du\, dv + \int_0^1\int_0^1 uv\, du\, dv = 0$$

By symmetry, all other pairs of integrals will vanish, so we get the result.

(6) Exercise 8.4.3

$W = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$. $F = \langle x, y, z\rangle$.

We want to verify:

$$\iiint_W \nabla\cdot F\, dV = \iint_{\partial W} F\cdot d\vec{S}$$

LHS: $\displaystyle \iiint_W \nabla\cdot F = \iiint_{x^2+y^2+z^2\le 1} \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\, dV = 3\iiint_{x^2+y^2+z^2\le 1} dV = 3\, \mathrm{Vol}(\text{unit sphere}) = \boxed{4\pi}$

RHS: $\displaystyle \iint_{\partial W} F\cdot d\vec{S} = \iint_S F\cdot \vec{n}\, dS = \iint_S \langle x,y,z\rangle\cdot\langle x,y,z\rangle\, dS = \iint_S x^2+y^2+z^2\, dS$

$\underset{\text{Unit sphere}}{=} \displaystyle\iint_S dS$ = Surface Area (Unit Sphere) $= \boxed{4\pi}$

Hence, LHS $= 4\pi =$ RHS, so we get the result