M312 - Fall 2013 - HW 13 - Enrique Areyan

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(1) Exercise 8.4.6. Let $\vec{F} = \langle x^3, y^3, z^3 \rangle$. Evaluate the surface integral of $\vec{F}$ over the unit sphere.

Solution: We want to compute $\iint_S \vec{F} d\vec{S}$. where S = unit sphere.

By Gauss' Divergence theorem:

$$\iint_{\partial S} \vec{F} \cdot d\vec{S} = \iiint_S (\nabla \cdot F) dV = \iiint_S 3(x^2+y^2+z^2)\, dV = 3 \iiint_S x^2+y^2+z^2\, dV.$$

S is not unit sphere!

But $x^2+y^2+z^2 = 1$ in the unit sphere. Therefore:

$$\iint_{\partial S} \vec{F} \cdot d\vec{S} = 3 \iiint_S dV = 3\left(\frac{4}{3}\pi\right) = \boxed{4\pi}$$

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(2) Exercise 8.4.7. Evaluate $\iint_{\partial W} \vec{F} \cdot d\vec{S}$, where $\vec{F} = \langle x, y, z \rangle$ and W is the unit cube (in the first octant). Perform the calculation directly and check by using the divergence theorem.

Solution: By Gauss' Divergence theorem:

$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W \operatorname{div} F\, dV = \int_0^1\int_0^1\int_0^1 3\, dV = \boxed{3}$$

Now, computing directly: (using the computations on example 2, page 462-463)

$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = -\iint_{S_1} F_3\, dS + \iint_{S_2} F_3\, dS - \iint_{S_3} F_1\, dS + \iint_{S_4} F_1\, dS - \iint_{S_5} F_2\, dS + \iint_{S_6} F_2\, dS,$$

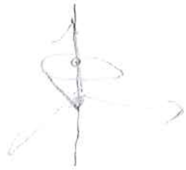
where $F_1 = x$, $F_2 = y$, $F_3 = z$.

$$= -\int_0^1\int_0^1 0\, dx\,dy + \int_0^1\int_0^1 1\, dx\,dy - \int_0^1\int_0^1 0\, dx\,dy + \int_0^1\int_0^1 1\, dx\,dy - \int_0^1\int_0^1 0\, dx\,dy + \int_0^1\int_0^1 1\, dx\,dy$$

$$= -0 + 1 - 0 + 1 - 0 + 1 = \boxed{3}$$

(3) Exercise 8.4.9. Let $\vec{F} = \langle y, z, xz \rangle$. Evaluate $\iint_{\partial W} \vec{F} \cdot d\vec{S}$ for each of the following regions W:

(a) $x^2+y^2 \le z \le 1$.



By divergence theorem:

$$\iint_{\partial W} \vec{F}\cdot d\vec{S} = \iiint_W \operatorname{div} F\, dV = \iiint_W x\, dV = \int_0^1 \iint_{x^2+y^2\le z} x\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^{2\pi} \int_0^z r^2\cos\theta\, dr\, d\theta\, dz = \int_0^1 \left[\overbrace{\int_0^{2\pi} \cos\theta\, d\theta}^{0} \left[\int_0^z r^2\, dr\right]\right] dz = \boxed{0}$$

(b) $x^2+y^2 \le z \le 1$ and $x \geq 0$.

Proceding as before.

$$\iint_{\partial W} \vec{F}\cdot d\vec{S} = \iint_{x^2+y^2\le 1} \int_{x^2+y^2}^{1} x\, dz\, dy\, dx = \iint_{x^2+y^2\le 1} x\left(1-(x^2+y^2)\right) dy\, dx$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^1 \left[r\cos\theta\,(1-r^2)\right] r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^1 \cos\theta\,(1-r^2)\, r^2\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^1 \cos\theta\, r^2 - \cos\theta\, r^4\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{\cos\theta\, r^3}{3} - \frac{\cos\theta\, r^5}{5}\right]_0^1 d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{3} - \frac{\cos\theta}{5}\, d\theta = \left[\frac{\sin\theta}{3} - \frac{\sin\theta}{5}\right]_{-\pi/2}^{\pi/2} = \left[\frac{\sin(\pi/2)}{3} - \frac{\sin(\pi/2)}{5}\right] - \left[\frac{\sin(-\pi/2)}{3} - \frac{\sin(-\pi/2)}{5}\right]$$

$$= \left(\frac{1}{3} - \frac{1}{5}\right) - \left(-\frac{1}{3} + \frac{1}{5}\right)$$

$$= \frac{2}{3} - \frac{2}{5} = \frac{10-6}{15} = \boxed{\frac{4}{15}}$$

(c) $x^2+y^2 \le z \le 1$ and $x \le 0$. Same as before, just changing the angle from $-\pi/2$ to $\pi/2 \rightarrow \pi/2 \rightsquigarrow 3\pi/2$.

$$\iint_{\partial W} \vec{F}\cdot d\vec{S} = \int_{\pi/2}^{3\pi/2} \int_0^1 \cos\theta(1-r^2)r^2\, dr\, d\theta = \left[\frac{\sin\theta}{3} - \frac{\sin\theta}{5}\right]_{\pi/2}^{3\pi/2}$$

$$= \left(\frac{\sin(3\pi/2)}{3} - \frac{\sin(3\pi/2)}{5}\right) - \left(\frac{\sin(\pi/2)}{3} - \frac{\sin(\pi/2)}{5}\right) = \left(-\frac{1}{3} + \frac{1}{5}\right) - \left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= -\frac{2}{3} + \frac{2}{5} = \frac{-10+6}{15} = \boxed{\frac{-4}{15}}$$

(4) Exercise 8.4.15. Evaluate $\iint_{\partial W} \vec{F}\cdot\vec{n}\, dA$, where, $F(x,y,z) = \langle x, y, -z\rangle$ and W is the unit cube in the first octant. Perform the calculation directly and check by using the divergene theorem.

Solution: By divergence theorem

$$\iint_{\partial W} \vec{F}\cdot\vec{n}\, dA = \iiint_W \operatorname{div}\vec{F}\, dV = \int_0^1\int_0^1\int_0^1 1\, dv = \boxed{1}$$

By direct computation: (using the computations on example 2, pages 462-463).

$$\iint_{\partial W}\vec{F}\cdot\vec{n}\,dA = -\int_0^1\int_0^1 -0\,dxdy + \int_0^1\int_0^1 -1\,dxdy - \int_0^1\int_0^1 0\,dxdy + \int_0^1\int_0^1 1\,dxdy$$

$$- \int_0^1\int_0^1 0\,dxdy + \int_0^1\int_0^1 1\,dxdy = 0-1-0+1-0+1 = \boxed{1}$$ ✓

(5) Exercise 8.4.16. Evaluate the surface integral $\iint_{\partial S} \vec{F}\cdot\vec{n}\, dA$, where $\vec{F}(x,y,z) = \langle 1, 1, z(x^2+y^2)^2\rangle$ and ∂S is the surface of the cylinder $x^2+y^2\le 1$, $0\le z\le 1$.

Solution: $$\iint_{\partial S}\vec{F}\cdot\vec{n}\,dA = \iiint_W \operatorname{div}F\,dV = \int_0^1\iint_{x^2+y^2\le 1}(x^2+y^2)^2\,dxdydz$$

$$= \int_0^1\int_0^{2\pi}\int_0^1 (r^2)^2 r\,dr\,d\theta\,dz = 2\pi\int_0^1 r^5\,dr = \frac{2\pi}{6}\left[r^6\right]_0^1 = \boxed{\frac{\pi}{3}}$$ ✓

(6) Exercise 8.5.3 Find dw in the following examples:

(a) $w = x^2y + y^3 \Rightarrow dw = d(x^2y+y^3) =$

$$= \frac{\partial}{\partial x}(x^2y+y^3)\,dx + \frac{\partial}{\partial y}(x^2y+y^3)\,dy$$

$$= \boxed{2xy\,dx + (x^2+3y^2)\,dy}$$ ✓

(b) $w = y^2\cos x\,dy + xy\,dx + dz \Rightarrow dw = d(y^2\cos x\,dy + xy\,dx + dz)$

$$= d(y^2\cos x\,dy) + d(xy\,dx) + \cancel{d(dz)}^{0} = d(y^2\cos x)\wedge dy + d(xy)\wedge dx$$

$$= (-y^2\sin x\,dx + 2y\cos x\,dy)\wedge dy + (y\,dx + x\,dy)\wedge dx$$

$$= -y^2\sin x\,dxdy + x\,dydx = x + y^2\sin x\,dydx = \boxed{-(x+y^2\sin x)\,dxdy}$$ ✓

(c) $w = xy\,dy + (x+y)^2 dx \Rightarrow dw = d(xy\,dy + (x+y)^2 dx)$

$= d(xy\,dy) + d((x+y)^2 dx) = (y\,dx + x\,dy) \wedge dy + (2(x+y)dx + 2(x+y)dy) \wedge dx$

$= y\,dxdy + 2(x+y)dydx = y - 2(x+y)\,dxdy = y - 2x - 2y\,dxdy = \boxed{-(2x+y)dxdy}$

(d) $w = x\,dxdy + z\,dydz + y\,dzdx \Rightarrow dw = d(x\,dxdy + z\,dydz + y\,dzdx)$

$= d(x\,dxdy) + d(z\,dydz) + d(y\,dzdx).$

$dx \wedge dxdy + dz \wedge dydz + dy \wedge dzdx = dydzdx = -dy\,dxdz = \boxed{dxdydz}$

(e) $w = (x^2+y^2)dydz \Rightarrow dw = d((x^2+y^2)dydz)$

$= (2x\,dx) \wedge dydz + (2y\,dy) \wedge dydz = \boxed{2x\,dxdydz}$

(f) $w = (x^2+y^2+z^2)\,dz \Rightarrow dw = d((x^2+y^2+z^2)dz)$

$= (2xdx + 2ydy + 2zdz) \wedge dz = \boxed{2xdxdz + 2ydydz}$

(g) $w = \dfrac{-x}{x^2+y^2} dx + \dfrac{y}{x^2+y^2} dy \Rightarrow dw = d\left(\dfrac{-x}{x^2+y^2} dx + \dfrac{y}{x^2+y^2} dy\right)$

$= \left[\dfrac{\partial}{\partial x}\left(\dfrac{-x}{x^2+y^2}\right) dx + \dfrac{\partial}{\partial y}\left(\dfrac{-x}{x^2+y^2}\right) dy\right] \wedge dx + \left[\dfrac{\partial}{\partial x}\left(\dfrac{y}{x^2+y^2}\right) dx + \dfrac{\partial}{\partial y}\left(\dfrac{y}{x^2+y^2}\right) dy\right] \wedge dy$

$= \dfrac{2xy}{(x^2+y^2)^2} dydx - \dfrac{2xy}{(x^2+y^2)^2} dxdy = \boxed{\dfrac{-4xy}{(x^2+y^2)^2} dxdy}$

(h) $w = x^2y\,dydz \Rightarrow dw = d(x^2y\,dydz)$

$= (2xy\,dx + x^2 dy) \wedge dydz = \boxed{2xy\,dxdydz}$

(7) Exercise 8.5.6. Let $V: K \to \mathbb{R}^3$ be a vector field defined by:

$V(x,y,z) = G(x,y,z)\,\hat{i} + H(x,y,z)\,\hat{j} + F(x,y,z)\,\hat{k}$, and

let η be the 2-form on K given by:

$$\eta = F\,dxdy + G\,dydz + H\,dzdx.$$

Show that $d\eta = (\operatorname{div} V)\,dx\,dy\,dz$.

Pf: $d\eta = d(F\,dxdy + G\,dydz + H\,dzdx) = \longrightarrow$

$$= \left(\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz\right)\wedge dxdy + \left(\frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy + \frac{\partial G}{\partial z}dz\right)\wedge dydz$$
$$+ \left(\frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy + \frac{\partial H}{\partial z}dz\right)\wedge dzdx$$

$$= \frac{\partial F}{\partial z}dzdxdy + \frac{\partial G}{\partial x}dxdydz + \frac{\partial H}{\partial y}dydzdx$$

$$= \frac{\partial F}{\partial z}dxdydz + \frac{\partial G}{\partial x}dxdydz + \frac{\partial H}{\partial y}dxdydz$$

$$= \left(\frac{\partial F}{\partial z} + \frac{\partial G}{\partial x} + \frac{\partial H}{\partial y}\right)dxdydz = (\operatorname{div} V)\,dxdydz$$

$$\Rightarrow \boxed{d\eta = (\operatorname{div} V)\,dxdydz}$$

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(8) Exercise 8.5.10. Let $w = (x+y)dz + (y+z)dx + (x+z)dy$, and let S be the upper part of the unit sphere; that is

$$S = \{(x,y,z) : x^2+y^2+z^2 = 1,\ z \geq 0\}.$$

∂S is the unit circle in the xy plane.

Evaluate $\int_{\partial S} w$ both directly and by Stoke's theorem.

Solution: By Stoke's theorem:

$\int_{\partial S} w = \int_S dw$; where $dw = d((x+y)dz + (y+z)dx + (x+z)dy)$

$$= (dx + dy)\wedge dz + (dy + dz)\wedge dx + (dx + dz)\wedge dy$$
$$= dxdz + dydz + dydx + dzdx + dxdy + dzdy$$
$$= (1-1)dxdz + (1-1)dydz + (1-1)dxdy = 0$$

$$\Rightarrow \int_{\partial S} w = \int_S dw = \int_S 0 = \boxed{0}$$

By direct computation:

$\int_{\partial S} w = \int_{\partial S} (x+y)dz + (y+z)dx + (x+z)dy$:

Using the parametrization for ∂w : $c(t) = (\cos t, \sin t)$ $0 \leq t \leq 2\pi$

$$\int_{\partial W} w = \int_0^{2\pi} -\sin^2 t + \cos^2 t \, dt = \int_0^{2\pi} \cos 2t \, dt = \left[\frac{\sin 2t}{2}\right]_0^{2\pi} = 0 - 0 = \boxed{0}$$