M312 - Enrique Areyan - Fall 2013 - HW 3

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(1) Prove the formula $\Delta(fg) = f\Delta g + 2\nabla f \cdot \nabla g + g\Delta f$.

Pf: let $f:\mathbb{R}^n \to \mathbb{R}$ and $g:\mathbb{R}^n \to \mathbb{R}$. then:

$$\Delta(fg) = \nabla \cdot \nabla(fg) = \nabla \cdot \left\langle \frac{\partial}{\partial x_1}(fg), \ldots, \frac{\partial}{\partial x_n}(fg) \right\rangle$$

$$= \nabla \cdot \left\langle \left(\frac{\partial}{\partial x_1} f\right) \cdot g + f\left(\frac{\partial}{\partial x_n} g\right), \ldots, \left(\frac{\partial}{\partial x_n} f\right) g + f\left(\frac{\partial}{\partial x_n} g\right) \right\rangle$$

$$= \frac{\partial}{\partial x_1}\left[\left(\frac{\partial}{\partial x_1} f\right) \cdot g + f\left(\frac{\partial}{\partial x_1} g\right)\right] + \cdots + \frac{\partial}{\partial x_n}\left[\left(\frac{\partial}{\partial x_n} f\right) \cdot g + f\left(\frac{\partial}{\partial x_n} g\right)\right]$$

$$= \frac{\partial}{\partial x_1}\left[\left(\frac{\partial}{\partial x_1} f\right) g\right] + \frac{\partial}{\partial x_1}\left[f\left(\frac{\partial}{\partial x_1} g\right)\right] + \cdots + \frac{\partial}{\partial x_n}\left[\left(\frac{\partial}{\partial x_n} f\right) \cdot g\right] + \frac{\partial}{\partial x_n}\left[f\left(\frac{\partial}{\partial x_n} g\right)\right]$$

$$= \left(\frac{\partial^2}{\partial x_1^2} f\right) \cdot g + \left(\frac{\partial}{\partial x_1} f\right)\left(\frac{\partial}{\partial x_1} g\right) + \left(\frac{\partial}{\partial x_1} f\right)\left(\frac{\partial}{\partial x_1} g\right) + f\left(\frac{\partial^2}{\partial x_1^2} g\right) + \cdots + \left(\frac{\partial^2}{\partial x_n^2} f\right) \cdot g + \left(\frac{\partial}{\partial x_n} f\right)\left(\frac{\partial}{\partial x_n} g\right) + \left(\frac{\partial}{\partial x_n} f\right)\left(\frac{\partial}{\partial x_n} g\right) + f\left(\frac{\partial^2}{\partial x_n^2} g\right)$$

$$= f\left(\frac{\partial^2}{\partial x_1^2} g + \cdots + \frac{\partial^2}{\partial x_n^2} g\right) + 2\left\langle \frac{\partial}{\partial x_1} f, \ldots, \frac{\partial}{\partial x_n} f \right\rangle \cdot \left\langle \frac{\partial}{\partial x_1} g, \ldots, \frac{\partial}{\partial x_n} g \right\rangle + g\left(\frac{\partial^2}{\partial x_1^2} f + \cdots + \frac{\partial^2}{\partial x_n^2} f\right)$$

$$= f\left[\left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle \cdot \left\langle \frac{\partial}{\partial x_1} g, \ldots, \frac{\partial}{\partial x_n} g \right\rangle\right] + 2\nabla f \cdot \nabla g + g\left[\left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle \cdot \left\langle \frac{\partial}{\partial x_1} f, \ldots, \frac{\partial}{\partial x_n} f \right\rangle\right]$$

$$= f[\nabla \cdot \nabla(g)] + 2\nabla f \cdot \nabla g + g[\nabla \cdot \nabla(f)]$$

$$= f\Delta g + 2\nabla f \cdot \nabla g + g\Delta f. \qquad \square$$

(2) Prove the formula $\operatorname{div}(f\nabla g - g\nabla f) = f\Delta g - g\Delta f$.

Pf: let $f:\mathbb{R}^n \to \mathbb{R}$ and $g:\mathbb{R}^n \to \mathbb{R}$. then:

$$\operatorname{div}(f\nabla g - g\nabla f) = \operatorname{div}\left(f\left\langle \frac{\partial}{\partial x_1} g, \ldots, \frac{\partial}{\partial x_n} g \right\rangle - g\left\langle \frac{\partial}{\partial x_1} f, \ldots, \frac{\partial}{\partial x_n} f \right\rangle\right)$$

$$= \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle \cdot \left\langle f\frac{\partial}{\partial x_1} g - g\frac{\partial}{\partial x_1} f, \ldots, f\frac{\partial}{\partial x_n} g - g\frac{\partial}{\partial x_n} f \right\rangle$$

$$= \frac{\partial}{\partial x_1}\left[f\frac{\partial}{\partial x_1} g - g\frac{\partial}{\partial x_1} f\right] + \cdots + \frac{\partial}{\partial x_n}\left[f\frac{\partial}{\partial x_n} g - g\frac{\partial}{\partial x_n} f\right]$$

$$= \frac{\partial}{\partial x_1}\left[f\frac{\partial}{\partial x_1} g\right] - \frac{\partial}{\partial x_1}\left[g\frac{\partial}{\partial x_1} f\right] + \cdots + \frac{\partial}{\partial x_n}\left[f\frac{\partial}{\partial x_n} g\right] - \frac{\partial}{\partial x_n}\left[g\frac{\partial}{\partial x_n} f\right]$$

$$= f\frac{\partial^2}{\partial x_1^2} g - g\frac{\partial^2}{\partial x_1^2} f + \cdots + f\frac{\partial^2}{\partial x_n^2} g - g\frac{\partial^2}{\partial x_n^2} f$$

$$= f\left[\frac{\partial^2}{\partial x_1^2} g + \cdots + \frac{\partial^2}{\partial x_n^2} g\right] - g\left[\frac{\partial^2}{\partial x_1^2} f + \cdots + \frac{\partial^2}{\partial x_n^2} f\right]$$

$$= f\Delta g - g\Delta f \qquad \square$$

3) For $(x,y) \in \mathbb{R}^2$, $(x,y) \neq (0,0)$, let $f(x,y) = \ln(x^2+y^2)$. Compute Δf.

olution:

$$\Delta f = \Delta(\ln(x^2+y^2)) = \nabla \cdot \nabla(\ln(x^2+y^2)) = \nabla \cdot \left\langle \frac{\partial}{\partial x}\ln(x^2+y^2), \frac{\partial}{\partial y}\ln(x^2+y^2) \right\rangle$$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \cdot \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle = \frac{\partial}{\partial x}\left[\frac{2x}{x^2+y^2}\right] + \frac{\partial}{\partial y}\left[\frac{2y}{x^2+y^2}\right]$$

$$= \frac{2}{x^2+y^2} + \frac{(2x)(-2x)}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} + \frac{(2y)(-2y)}{(x^2+y^2)^2} = \frac{2(x^2+y^2) - 4x^2 + 2(x^2+y^2) - 4y^2}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4x^2+2x^2+2y^2-4y^2}{(x^2+y^2)^2} = \frac{4x^2+4y^2-4x^2-4y^2}{(x^2+y^2)^2} = \frac{0}{(x^2+y^2)^2} = \boxed{0}$$

) Show that for any $v, w \in \mathbb{R}^3$ one has: $\sqrt{\|v\|^2\|w\|^2 - (v \cdot w)^2} = \|v \times w\|$

: Let $v, w \in \mathbb{R}^3$ be $v = \langle v_1, v_2 \rangle$ and $w = \langle w_1, w_2 \rangle$. Let us first compute he left-hand side then compute the right-hand size and prove that these uantities are equal. So, on the one hand:

$$\sqrt{\|v\|^2\|w\|^2 - (v \cdot w)^2} = \sqrt{(v_1^2+v_2^2+v_3^2)(w_1^2+w_2^2+w_3^2) - (v_1w_1+v_2w_2+v_3w_3)^2}$$

$$v_1^2w_1^2 + v_1^2w_2^2 + v_1^2w_3^2 + v_2^2w_1^2 + v_2^2w_2^2 + v_2^2w_3^2 + v_3^2w_1^2 + v_3^2w_2^2 + v_3^2w_3^2 - \left[(v_1w_1+v_2w_2)^2 + 2(v_1w_1+v_2w_2)v_3w_3 + v_3^2w_3^2\right]$$

$$\cancel{v_1^2w_1^2} + v_1^2w_2^2 + v_1^2w_3^2 + v_2^2w_1^2 + \cancel{v_2^2w_2^2} + v_2^2w_3^2 + v_3^2w_1^2 + v_3^2w_2^2 + \cancel{v_3^2w_3^2} - \cancel{v_1^2w_1^2} - 2v_1w_1v_2w_2 - \cancel{v_2^2w_2^2} - 2v_1w_1v_3w_3 - 2v_2w_2v_3w_3 - \cancel{v_3^2w_3^2}$$

$$v_2^2w_3^2 - 2v_2v_3w_2w_3 + v_3^2w_2^2 + v_1^2w_3^2 - 2v_1v_3w_1w_3 + v_3^2w_1^2 + v_1^2w_2^2 - 2v_1v_2w_1w_2 + v_2^2w_1^2 \stackrel{!}{=} (1)$$

n the other:

$$v \times w = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \hat{\imath}(v_2w_3 - v_3w_2) - \hat{\jmath}(v_1w_3 - v_3w_1) + \hat{k}(v_1w_2 - v_2w_1)$$

$$= \langle v_2w_3 - v_3w_2, v_1w_3 - v_3w_1, v_1w_2 - v_2w_1 \rangle. \text{ therefore}$$

$$\|v \times w\| = \sqrt{(v_2w_3 - v_3w_2)^2 + (v_1w_3 - v_3w_1)^2 + (v_1w_2 - v_2w_1)^2}$$

$$v_2^2w_3^2 - 2v_2v_3w_2w_3 + v_1^2w_3^2 - 2v_1v_3w_1w_3 + v_3^2w_1^2 + v_1^2w_2^2 - 2v_1v_2w_1w_2 + v_2^2w_1^2 = (2)$$

nce (1) = (2) we have show that $\sqrt{\|v\|^2\|w\|^2 - (v \cdot w)^2} = \|v \times w\|$

(5) Compute the curvature of the path $c(t) = \langle \cos t, \sin t, t^2 \rangle$ at arbitrary t.

Solution: Since this path is in $\mathbb{R}^3$ we can compute its curvature $K(t)$ as

$$K(t) = \frac{\| c'(t) \times c''(t) \|}{\| c'(t) \|^3}.$$

Compute each piece:

$c(t) = \langle \cos t, \sin t, t^2 \rangle \Rightarrow c'(t) = \langle -\sin t, \cos t, 2t \rangle \Rightarrow c''(t) = \langle -\cos t, -\sin t, 2 \rangle$

$$c'(t) \times c''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin t & \cos t & 2t \\ -\cos t & -\sin t & 2 \end{vmatrix} = \hat{i}(2\cos t + 2t\sin t) - \hat{j}(-2\sin t + 2t\cos t) + \hat{k}(\sin^2 t + \cos^2 t).$$

then,

therefore, $c'(t) \times c''(t) = \langle 2\cos t + 2t\sin t, 2\sin t - 2t\cos t, 1 \rangle$

$$\| c'(t) \times c''(t) \| = \sqrt{(2\cos t + 2t\sin t)^2 + (2\sin t - 2t\cos t)^2 + 1^2}$$

$$= \sqrt{4\cos^2 t + \cancel{8t\cos t\sin t} + 4t^2\sin^2 t + 4\sin^2 t - \cancel{8t\sin t\cos t} + 4t^2\cos^2 t + 1}$$

$$= \sqrt{4\cancel{(\sin^2 t + \cos^2 t)} + 4t^2\cancel{(\sin^2 t + \cos^2 t)} + 1} = \sqrt{4t^2 + 5}$$

Also, $\| c'(t) \| = \sqrt{\sin^2 t + \cos^2 t + 4t^2} = \sqrt{4t^2 + 1}$. Hence, the curvature is:

$$K(t) = \frac{\| c'(t) \times c''(t) \|}{\| c'(t) \|^3} = \frac{\sqrt{4t^2+5}}{\left(\sqrt{4t^2+1}\right)^3} = \left[\frac{4t^2+5}{(4t^2+1)^3}\right]^{1/2}$$

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(6) Find an appropiate parametrization for the curve which is the interse[ction] of the surfaces $y = x$ and $z = x^2$ from the point $(-2,-2,4)$ and $(1,1,1)$. Find the total curvature of this curve.

Solution: Let $c(t)$ be a parametrization of the curve, $c(t) = \langle x(t), y(t), \ldots$ Let $x(t) = t$. then $y(t) = x(t) = t$ and $z(t) = t^2$. A parametrization then would be: $c(t) = \langle t, t, t^2 \rangle$. Our limits become:

$t = -2 \Rightarrow c(-2) = \langle -2, -2, 4 \rangle$

$t = 1 \Rightarrow c(1) = \langle 1, 1, 1 \rangle$

Hence, our total curvature will be: $\int_C K\,ds$, where $K(t)$ is the curvatur[e] o compute the curvature we need:

$c(t) = \langle t, t, t^2 \rangle \Rightarrow c'(t) = \langle 1, 1, 2t \rangle \Rightarrow c''(t) = \langle 0, 0, 2 \rangle.$

$$c'(t) \times c''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = \hat{i}(2) - \hat{j}(2) + \hat{k}(0) = \langle 2, -2, 0 \rangle.$$

$\|c'(t) \times c''(t)\| = \sqrt{2^2 + (-2)^2 + 0^2} = \sqrt{8} = 2\sqrt{2}$. Also,

$\|c'(t)\| = \sqrt{1^2 + 1^2 + (2t)^2} = \sqrt{4t^2 + 2}$. therefore, the curvature K is:

$$\boxed{K(t) = \frac{2\sqrt{2}}{(4t^2+2)^{3/2}}}$$

e total curvature is given by:

$$\int_C K\,ds = \int_{-2}^{1} K\,\|c'(t)\|\,dt = \int_{-2}^{1} \frac{2\sqrt{2}}{(4t^2+2)^{3/2}} (4t^2+2)^{1/2}\,dt = 2\sqrt{2}\int_{-2}^{1} \frac{1}{4t^2+2}\,dt = 2\sqrt{2}\int_{-2}^{1} \frac{1}{2(2t^2+1)}\,dt$$

$$\sqrt{2}\int_{-2}^{1} \frac{1}{2t^2+1}\,dt = \sqrt{2}\left[\frac{\arctan(\sqrt{2}x)}{\sqrt{2}}\right]_{-2}^{1} = \arctan(\sqrt{2}) - \arctan(-2\sqrt{2}) \approx \boxed{2.186276604 \text{ rad}}$$

) SHOW that the work done by the gravitational vector field in $\mathbb{R}^3$ centered the origin (with $G = m = M = 1$) as a particle moves from point p to point depends only on $\|p\|$ and $\|q\|$.

: this follows from the fact that the work done by a gradient ctor field does not depend on the path, only depends on the start d end points.

e gravitational vector field $\vec{F}$ given by $\vec{F} = -\frac{mMG}{r^3}\vec{r}$, where $(x, y, z) = (x, y, z)$ and $r = \|\vec{r}\|$, is actually a gradient field with avitational potential $V = -\frac{mMG}{r}$, i.e., $\vec{F} = -\nabla V$. Now, take $= m = M = 1$ to get $\vec{F} = -\nabla\left(-\frac{1}{r}\right)$.

definition, the work done by this gradient field is

$$W = \int_a^b \vec{F}(c(t)) \cdot c'(t)\,dt = -\int_a^b \nabla\left(-\frac{1}{r}\right) \cdot c'(t)\,dt = -\int_a^b \frac{d}{dt}\left(V(c(t))\right)dt$$

$= V(c(a)) - V(c(b))$, where a is such that $c(a) = p$ and b is such that $c(b) = q$. But then,

$= V(p) - V(q) = -\frac{1}{\|p\|} + \frac{1}{\|q\|} = \boxed{\frac{1}{\|q\|} - \frac{1}{\|p\|}}$ only depends on $\|p\|$ and $\|q\|$

(8) Compute $\int_C \frac{x\,dx + y\,dy}{x^2+y^2}$, where $c(t) = (e^t, t^2)$, $0 \leq t \leq 1$.

Solution: $\int_C \frac{x\,dx + y\,dy}{x^2+y^2} = \int_C \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle \cdot \langle dx, dy \rangle = \int_C \vec{F} \cdot d\vec{s}$, where

$\vec{F} = \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$. the key observation here is that $\vec{F}$ is a gradient field since:

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{-2xy}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = \frac{\partial F_2}{\partial y}$$

So, let us find its potential function $f: \mathbb{R}^2 \to \mathbb{R}$, s.t., $\vec{F} = \nabla f$.

We know that f must satisfy:

$$\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2} \Rightarrow f = \int \frac{\partial f}{\partial x} \partial x = \int \frac{x}{x^2+y^2} \partial x$$

; make the change: $u = x^2+y^2$ $\frac{\partial u}{\partial x} = 2x\,\partial x$ $\Rightarrow \frac{\partial u}{2\,\partial x} = x\,\partial x$

$$\rightarrow = \int \frac{1}{u} \frac{\partial u}{2\partial x} = \frac{1}{2}\int \frac{1}{u}\frac{\partial u}{\partial x}$$

$= \frac{1}{2}[\ln(u)] + g(y)$; where $g(y)$ is a pure function of y.

$= \frac{1}{2}[\ln(x^2+y^2)] + g(y) = f(x,y)$.

Also,

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\frac{1}{2}[\ln(x^2+y^2)] + g(y)\right] = \frac{1}{2}\left[\frac{2y}{x^2+y^2}\right] + g'(y) = \frac{y}{x^2+y^2} + g'(y) = \frac{y}{x^2+y^2}$$

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = c$, for c a constant.

therefore, our potential function is given by:

$$\boxed{f(x,y) = \frac{\ln(x^2+y^2)}{2} + c}$$

Now we can evaluate the line integral

$$\int_C F \cdot d\vec{s} = \int_C \nabla f \cdot d\vec{s} = f(c(1)) - f(c(0)) = f(e^1, 1) - f(1,0) = (\ln(e^2+1) - \ln(1))/2$$

$$= \frac{\ln(e^2+1)}{2} \simeq \boxed{1.06346400552}$$