M312 - Fall 2013 - HW6 - Enrique Areyan

100 + 5

(1) Exercise 6.1.2. Determine if the following functions $T: \mathbb{R}^2 \to \mathbb{R}^2$ are one-to-one and/or onto.

Solution:

(a) $T(x,y,z) = (2x+y+3z,\ 3y-4z,\ 5x)$. this is a linear transformation:

$T(\alpha(x_1,x_2,x_3) + \beta(y_1,y_2,y_3)) = T(\alpha x_1+\beta y_1, \alpha x_2+\beta y_2, \alpha x_3+\beta y_3)$

$= (2(\alpha x_1+\beta y_1)+\alpha x_2+\beta y_2+3(\alpha x_3+\beta y_3),\ 3(\alpha x_2+\beta y_2)-4(\alpha x_3+\beta y_3),\ 5(\alpha x_1+\beta y_1))$

$= ((2\alpha x_1+\alpha x_2+3\alpha x_3)+(2\beta y_1+\beta y_2+3\beta y_3),\ (3\alpha x_2-4\alpha x_3)+(3\beta y_2-4\beta y_3),\ (5\alpha x_1+5\beta y_1))$

$= (2\alpha x_1+\alpha x_2+3\alpha x_3,\ 3\alpha x_2-4\alpha x_3,\ 5\alpha x_1) + (2\beta y_1+\beta y_2+3\beta y_3,\ 3\beta y_2-4\beta y_3,\ 5\beta y_1)$

$= \alpha(2x_1+x_2+3x_3,\ 3x_2-4x_3,\ 5x_1) + \beta(2y_1+y_2+3y_3,\ 3y_2-4y_3,\ 5y_1)$

$= \alpha T(x_1,x_2,x_3) + \beta T(y_1,y_2,y_3)$. So we can compute the matrix A associated to T.

$T(1,0,0) = (2,0,5)$; $T(0,1,0) = (1,3,0)$; $T(0,0,1) = (3,-4,0)$

$\Rightarrow A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & -4 \\ 5 & 0 & 0 \end{bmatrix}$. Let us compute the determinant of A

$\det(A) = 2\begin{vmatrix} 3 & -4 \\ 0 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & -4 \\ 5 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 3 \\ 5 & 0 \end{vmatrix} = 0 - 1(20) + 3(-15) = -20 - 45 = -65 \neq 0$

$\Rightarrow T$ is one-to-one and onto.

(b) $T(x,y,z) = (y\sin x,\ z\cos y,\ xy)$. This is NOT a linear transformation:

$T((1,1,0)+(0,0,1)) = T(1,1,1) = (\sin(1), \cos(1), 1) \neq (\sin(1),1,1) = (\sin(1),0,1) + (0,1,0)$

$= T(1,1,0) + T(0,0,1)$. So let us check one-to-one and onto separately:

1-1: T is not 1-1. Consider $(1,0,0) \in \mathbb{R}^3$, $(0,1,0) \in \mathbb{R}^3$. Clearly $(1,0,0) \neq (0,1,0)$

But $T(1,0,0) = (0,0,0) = T(0,1,0)$.

onto: T is not onto. since $(1,0,0)$ has no preimage under T. Suppose it does. then, there exists $(x_1,x_2,x_3) \in \mathbb{R}^3$ such that $T(x_1,x_2,x_3) = (1,0,0)$

$\Leftrightarrow (x_2\sin(x_1),\ x_3\cos(x_2),\ x_1x_2) = (1,0,0)$

$\Leftrightarrow \begin{cases} x_2\sin(x_1) = 1 \\ x_3\cos(x_2) = 0 \\ x_1x_2 = 0 \end{cases}$ From the third equation we get that $x_1 = 0$ OR $x_2 = 0$

If $x_1 = 0$ then; $x_2\sin(0) = 1 \Rightarrow 0 = 1$; contradiction.

If $x_2 = 0$ then; $x_2\sin(0) = 1 \Rightarrow 0 = 1$; contradiction.

In any case we get a contradiction, so there exists no such $(x_1,x_2,x_3) \in \mathbb{R}^3$

therefore, T is not onto.

(c) $T(x,y,z) = (xy, yz, xz)$. T is not a linear map since:

$T((1,1,0)+(0,0,1)) = T(1,1,1) = (1,1,1) \neq (1,0,0) = (1,0,0)+(0,0,0) = T(1,1,0)+T(0,0,1)$

Let us check one-to-one and onto separately:

1-1: T is not one-to-one. Consider $(1,0,0) \in \mathbb{R}^3$ and $(0,1,0) \in \mathbb{R}^3$. clearly $(1,0,0) \neq (0,1,0)$, but $T(1,0,0) = (0,0,0) = T(0,1,0)$.

onto: T is not onto since $(1,1,0) \in \mathbb{R}^3$ has no preimage under T. Suppose it does. then, there exists $(x_1,x_2,x_3) \in \mathbb{R}^3$ s.t. $T(x_1,x_2,x_3) = (1,1,0) \Leftrightarrow$

$(x_1x_2, x_2x_3, x_1x_3) = (1,1,0) \Rightarrow \begin{cases} x_1x_2 = 1 \\ x_2x_3 = 1 \\ x_1x_3 = 0 \end{cases}$ From last eq we get $x_1 = 0$ or $x_3 = 0$.

If $x_1 = 0 \Rightarrow x_1x_2 = 1 \Rightarrow 0 = 1$ Contradiction.

If $x_3 = 0 \Rightarrow x_2x_3 = 1 \Rightarrow 0 = 1$ Contradiction

In any case we get a contradiction

therefore, there exists no such $(x_1,x_2,x_3) \in \mathbb{R}^3$. Hence, T is not onto

(d) $T(x,y,z) = (e^x, e^y, e^z)$. T is not a linear map since

$T((1,1,0)+(0,0,1)) = T(1,1,1) = (e,e,e) \neq (e+1, e+1, e+1) = (e,e,1)+(1,1,e) = T(1,1,0)+T(0,0,1)$

Let us check one-to-one and onto separately:

1-1: T is one-to-one. Let $(x_1,x_2,x_3) \in \mathbb{R}^3$, $(y_1,y_2,y_3) \in \mathbb{R}^3$ and suppose that $T(x_1,x_2,x_3) = T(y_1,y_2,y_3)$. then $(e^{x_1}, e^{x_2}, e^{x_3}) = (e^{y_1}, e^{y_2}, e^{y_3}) \Leftrightarrow$

$\left.\begin{cases} e^{x_1} = e^{y_1} \Rightarrow x_1 = y_1 \\ e^{x_2} = e^{y_2} \Rightarrow x_2 = y_2 \\ e^{x_3} = e^{y_3} \Rightarrow x_3 = y_3 \end{cases}\right\}$ $(x_1,x_2,x_3) = (y_1,y_2,y_3)$.

onto: T is not onto, since $(-1,-1,-1) \in \mathbb{R}^3$ has no preimage under T. Suppose that it does. then, there exists $(x_1,x_2,x_3) \in \mathbb{R}^3$ s.t $T(x_1,x_2,x_3) = (-1,-1,-1)$

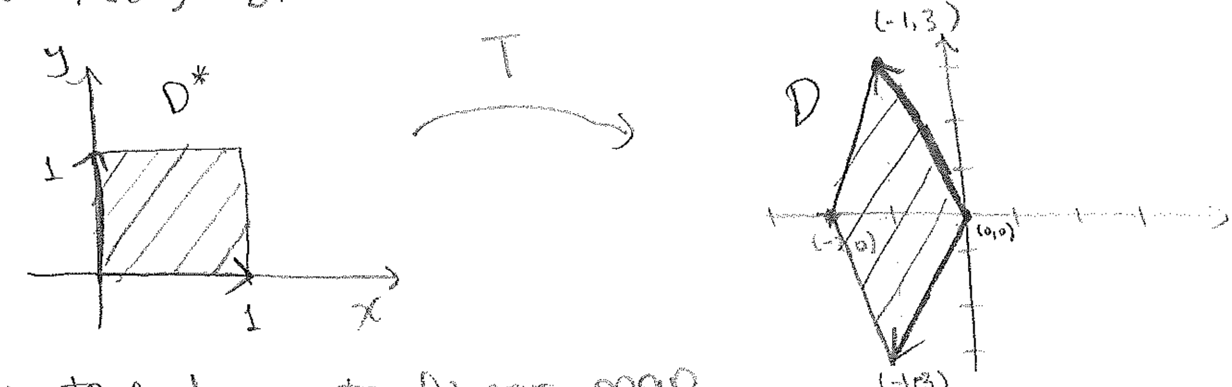
$\Leftrightarrow (e^{x_1}, e^{x_2}, e^{x_3}) = (-1,-1,-1) \Rightarrow \begin{cases} e^{x_1} = -1 \\ e^{x_2} = -1 \\ e^{x_3} = -1 \end{cases}$, but e^x is a positive function, so these eqs. cannot be satisfied.

therefore, there exists no such $(x_1,x_2,x_3) \in \mathbb{R}^3$.

Hence, T is not onto.

(2) Exercise 6.1.4 Let D be a parallelogram with vertices $(0,0)$, $(-1,3)$, $(-2,0)$, $(-1,-3)$. Let $D^* = [0,1] \times [0,1]$. Find a linear map T such that $T(D^*) = D$.

Solution



It suffices to find an onto linear map since linear maps preserve edges, so parallelograms are map to parallelograms. Moreover, we just need to send the two vectors that span D^* to two vectors that span D, i.e.,

$$T(1,0) = (-1,3) \quad \text{and} \quad T(0,1) = (-1,-3).$$

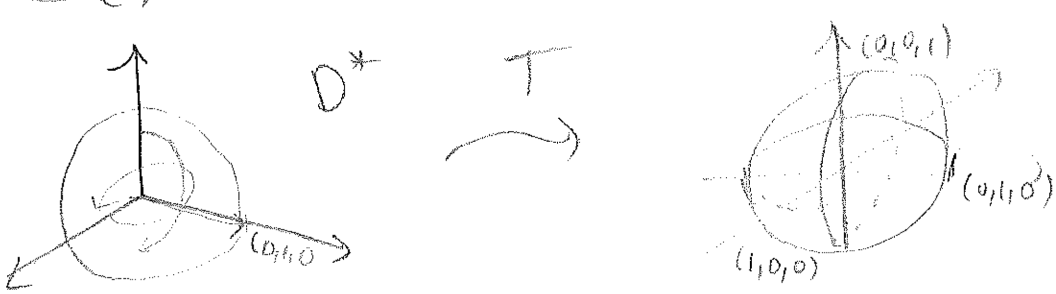
But T is linear so we can form its matrix: $T(x,y) = \begin{bmatrix} -1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. So the linear map is $T(x,y) = (-x-y,\ 3x-3y)$. Since the determinant of the matrix associated with T is $\det\left(\begin{bmatrix} -1 & -1 \\ 3 & -3 \end{bmatrix}\right) = 3+3 = 6 \neq 0$, we know that T is invertible and hence, onto.

(3) Exercise 6.1.11. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the spherical coordinate mapping defined by $(\rho, \phi, \theta) \mapsto (x,y,z)$ where

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

Let D^* be the set of points (ρ, ϕ, θ) such that: $\phi \in [0,\pi]$, $\theta \in [0,2\pi)$, $\rho \in [0,1]$. (i) Find $D = T(D^*)$.

Solution: (i) A sketch for the region is



Therefore, the region D^* gets map into the unit ball centered at the origin.

$$D = T(D^*) = \{ (x,y,z) \mid x^2+y^2+z^2 \leq 1 \}.$$

However, as it stands, T is not one-to-one because

$(0,0,0) \in \mathbb{R}^3$, $(0,\pi,2\pi) \in \mathbb{R}^3$ are such that

$T(0,0,0) = (0,0,0) = T(0,\pi,2\pi)$; so we have to restrict T to be on the subset of D^* given by $(0,1] \times (0,\pi) \times (0,2\pi]$ to make it into a one-to-one function.

(4) Exercise 6.2.3. Let D be the unit disk : $x^2+y^2 \leq 1$. Evaluate

$$\iint_D e^{x^2+y^2}\,dx\,dy$$ by making a change of variables to polar coordinates.

Solution: Polar coordinates are given by $x = r\cos\theta$; $y = r\sin\theta$; $r^2 = x^2+y^2$

therefore $$\iint_D e^{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_0^1 e^{r^2}\, r\,dr\,d\theta \ ;$$ changing: $u = e^{r^2} \Rightarrow du = 2re^{r^2}dr \Rightarrow e^{r^2} r\,dr = du/2$

$$\rightsquigarrow \int_0^{2\pi}\int_{u_0}^{u_1} \frac{du}{2}\,d\theta = \frac{1}{2}\int_0^{2\pi} [u]_{u_0}^{u_1}\,d\theta \rightsquigarrow \frac{1}{2}\int_0^{2\pi} \left[e^{r^2}\right]_0^1 d\theta = \frac{1}{2}\int_0^{2\pi} e-1\,d\theta$$

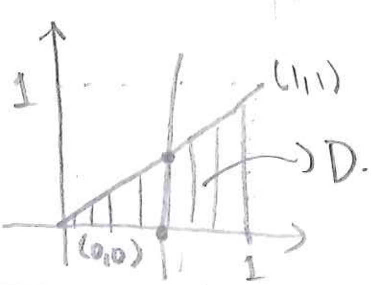
$$= \frac{e-1}{2}(2\pi) = \boxed{\pi(e-1)}$$

(5) Exercise 6.2.4. Let D be the region $0 \leq y \leq x$ and $0 \leq x \leq 1$.
Evaluate $\iint_D (x+y)\,dx\,dy$ by making the change of variables

$$x = u+v \ , \quad y = u-v.$$

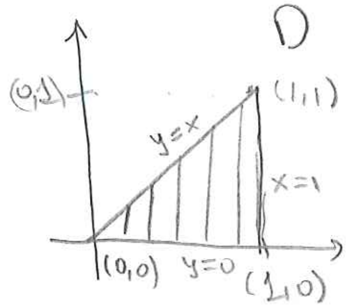
Solution : First, let us compute this integral directly by using an iterated integral :

$$\iint_D (x+y)\,dx\,dy = \int_0^1\int_0^x x+y\,dy\,dx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^x dx$$



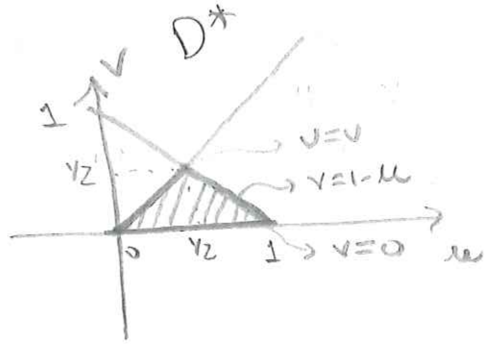
$$= \int_0^1 x^2 + \frac{x^2}{2}\,dx = \frac{3}{2}\int_0^1 x^2\,dx = \frac{1}{2}\left[x^3\right]_0^1 = \boxed{\frac{1}{2}}$$

Now, let us do the change of variables:



D is bounded by

$y=0$, $x=1$, $y=x$ } In the uv plane $\longrightarrow$ $u=v$, $v=1-u$, $v=0$



Compute the Jacobian; $J=\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1=-2.$

then

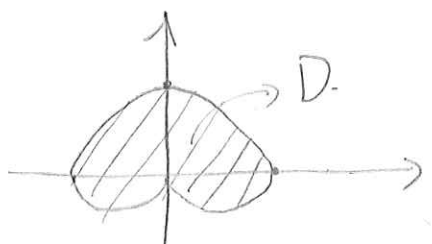
$$\iint_D x+y\,dxdy = \iint_{D^*} u+v+u-v\,|J|\,dudv = \iint_{D^*} 2u\,|-2|\,dudv = 4\iint_{D^*} u\,dudv$$

Hence, $4\iint_{D^*} u\,dudv = 4\int_0^{1/2}\int_v^{1-v} u\,dudv = \frac{4}{2}\int_0^{1/2}[u^2]_v^{1-v}\,dv = 2\int_0^{1/2}(1-v)^2-v^2\,dv$

$= 2\int_0^{1/2} 1-2v+\cancel{v^2}-\cancel{v^2}\,dv = 2\int_0^{1/2}1-2v\,dv = 2[v-v^2]_0^{1/2} = 2[\frac{1}{2}-\frac{1}{4}] = 2\,\frac{1}{4} = \boxed{\frac{1}{2}}$; so the integral via change of variable agrees with the direct integral.

(6) Exercise 6.2.13. Use double integrals to find the area inside the curve $r = 1+\sin\theta$.

Solution:



WANT TO compute $\iint_D dA$.

$$\int_0^{2\pi}\int_0^{1+\sin\theta} r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^{1+\sin\theta} dr\,d\theta = \frac{1}{2}\int_0^{2\pi}(1+\sin\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}1+2\sin\theta+\sin^2\theta\,d\theta$$

$$\frac{1}{2}\left[\theta-2\cos\theta+\frac{1}{2}(\theta-\sin\theta\cos\theta)\right]_0^{2\pi} = \frac{1}{2}\left[\left(2\pi-2+\frac{1}{2}(2\pi)\right)-\left(0-2+\frac{1}{2}(0-0)\right)\right]$$

$$=\frac{1}{2}[2\pi-2+\pi+2] = \boxed{\frac{3\pi}{2}}$$

(7) Exercise 6.2.17 Using Polar Coordinates, find the area bounded by the lemniscate $(x^2+y^2)^2 = 2a^2(x^2-y^2)$

Solution: First, let us write this equation in polar coordinates

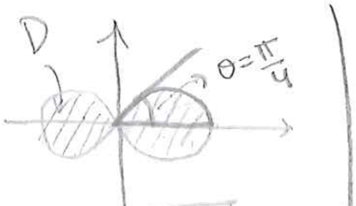
$$(x^2+y^2)^2 = 2a^2(x^2-y^2) \rightsquigarrow r^4 = 2a^2(r^2(\cos^2\theta - \sin^2\theta))$$

only positive b/c $r \geq 0$

$$\iff r^2 = 2a^2(\cos^2\theta - \sin^2\theta) \iff r^2 = 2a^2(\cos^2\theta - 1 + \cos^2\theta)$$

$$\iff r^2 = 2a^2(2\cos^2\theta - 1) \iff r^2 = 2a^2\cos(2\theta) \iff r = \sqrt{2a^2\cos(2\theta)}$$

$$\iff r = a\sqrt{2\cos(2\theta)}$$



$r = 0 \Rightarrow \boxed{\theta = \frac{\pi}{4}} + 2k$

$$\iint_D dA = 4\int_0^{\pi/4}\int_0^{\sqrt{2a^2\cos(2\theta)}} r\,dr\,d\theta = 2\int_0^{\pi/4} [r^2]_0^{\sqrt{2a^2\cos(2\theta)}}\,d\theta$$

$$= 2\int_0^{\pi/4} a^2\cos(2\theta)\,d\theta = 2a^2\int_0^{\pi/4}\cos(2\theta)\,d\theta = 2a^2[\sin(2\theta)]_0^{\pi/4}$$

$$= 2a^2\left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right] = \boxed{2a^2}$$

(8) Exercise 6.2.25

Evaluate $\iiint_W \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}\,dV$, where W is the solid bounded by the two spheres $x^2+y^2+z^2 = a^2$ and $x^2+y^2+z^2 = b^2$, where $0 < b < a$

Solution: Changing to spherical coordinates.

$$\int_b^a\int_0^\pi\int_0^{2\pi} \frac{\cancel{\rho^2}\sin\varphi}{\cancel{\rho^3}}\,d\theta\,d\varphi\,d\rho = 2\pi\int_b^a\int_0^\pi \frac{\sin\varphi}{\rho}\,d\varphi\,d\rho = 2\pi\int_b^a \frac{1}{\rho}\int_0^\pi \sin\varphi\,d\varphi\,d\rho$$

$$= 2\pi\int_b^a \frac{1}{\rho}[-\cos\varphi]_0^\pi\,d\rho = 2\pi\int_b^a \frac{1}{\rho}[1+1]\,d\rho = 4\pi\int_b^a \frac{1}{\rho}\,d\rho = 4\pi[\ln(\rho)]_b^a$$

$$= \boxed{4\pi[\ln(a) - \ln(b)]}$$

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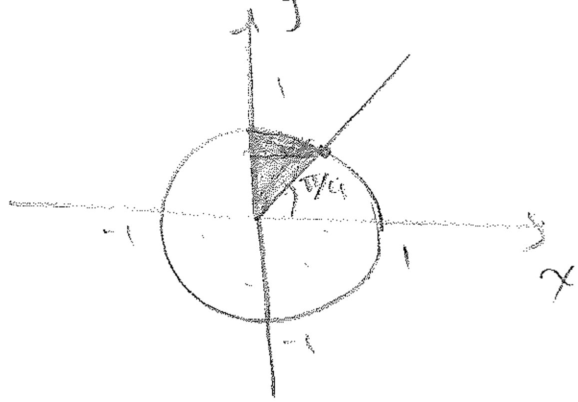
(9) Exercise 6.2.28. Evaluate $\iint_D x^2\,dx\,dy$, where D is determined by the two conditions: $0 \le x \le y$ and $x^2+y^2 \le 1$.

Solution:



$$\iint_D x^2\,dx\,dy \leadsto_{\text{changing to polar}} \int_{\pi/4}^{\pi/2}\int_0^1 r^2\cos^2\theta\; r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\cos^2\theta\,d\theta \int_0^1 r^3\,dr$$

$$= \left[\frac{1}{2}(\theta + \sin\theta\cos\theta)\right]_{\pi/4}^{\pi/2}\left[\frac{r^4}{4}\right]_0^1$$

$$= \left[\frac{1}{2}\left(\frac{\pi}{2}+0\right) - \frac{1}{2}\left(\frac{\pi}{4}+\sin(\pi/4)\cos(\pi/4)\right)\right]\frac{1}{4}$$

$$= \left[\frac{\pi}{4} - \frac{1}{2}\left(\frac{\pi}{4}+\frac{2}{4}\right)\right]\frac{1}{4}$$

$$= \left[\frac{\pi}{4} - \frac{1}{2}\left(\frac{2+\pi}{4}\right)\right]\frac{1}{4}$$

$$= \left[\frac{\pi}{4} - \frac{2+\pi}{8}\right]\frac{1}{4}$$

$$= \frac{\pi}{16} - \frac{2+\pi}{32} = \frac{2\pi-2-\pi}{32} = \boxed{\frac{\pi-2}{32}}$$

(10) Compute the volume of the following set:

$$W = \{(x,y,z) \in \mathbb{R}^3 : 0 \le z \le 1, (x^2+y^2+z^2-2yz)^2 \le (1-z)^2(x^2-y^2-z^2+2yz)\}.$$

Solution: We can compute the volume of W as

$$\iiint_W dx\,dy\,dz.$$

For the domain in the xy-plane: $z=0$, we get

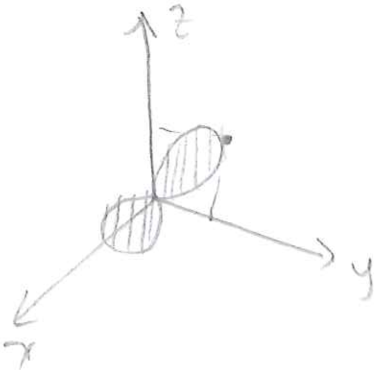
$(x^2+y^2)^2 \le (x^2-y^2)$; this is a lemniscate with parameter

$2a^2 = 1 \Rightarrow \boxed{a = \frac{1}{\pm\sqrt{2}}}$

Now, If we set $z=1$, we get

$(x^2+y^2+1-2y)^2 \le 0 \iff x^2+y^2+1-2y \le 0$

$\iff x^2+(y-1)^2-1+1 \le 0 \iff x^2+(y-1)^2 \le 0 \Rightarrow \boxed{x=0\ ;\ y=1}$



By Cavalieri principle & Ex. 7

$A(\text{leminscate at } z=0) = 2\left(\frac{1}{\pm\sqrt{2}}\right)^2 = \boxed{1}$

If we can find the area of the object by cross section of z, i.e. setting $z=c$, c a constant, then by Cavaliery

$Vol(W) = \int_0^1 A(z)\,dz.$; But we know $A(0) = 1$; $A(1) = 0$

Then?

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