M343 Homework 5

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Section 3.4

12. Consider the homogeneous, 2nd O.D.E with constant coefficients: $y'' - 6y' + 9y = 0$ and initial conditions: $y(0) = 0, y'(0) = 2$. The characteristic equation of this O.D.E is $(r-3)^2 = 0$, so we have two repeated real roots $r_1 = r_2 = 3$. The solution is given by $y(t) = C_1y_1 + C_2y_2$, where $y_1 = e^{3t}$ and $y_2 = te^{3t}$ (obtained by reduction of order). So, the general solution is:

$$y(t) = C_1e^{3t} + C_2te^{3t}$$

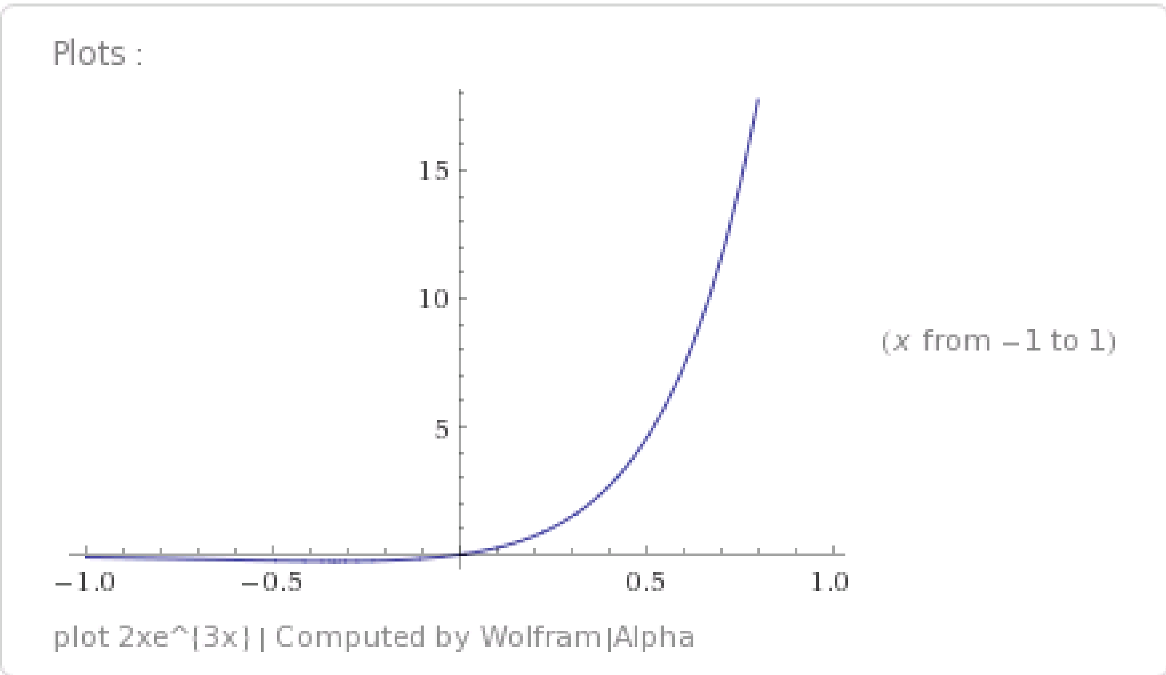
Solving for the constants C_1, C_2 using the initial conditions we obtain:

$$\begin{cases} y(0) = 0 = C_1 \cdot e^0 + C_2 \cdot 0 \cdot e^0 \Longrightarrow C_1 = 0 \\ \\ y'(0) = 2 = C_2[e^0 + 3 \cdot 0 \cdot e^0] \Longrightarrow C_2 = 2 \end{cases}$$

The solution for the the I.V.P is:

$$\boxed{y(t) = 2te^{3t}}$$

The graph for this solution is:



Also, $\lim_{t\to\infty} y(t) = \infty$

14. Consider the homogeneous, 2nd O.D.E with constant coefficients: $y'' + 4y' + 4y = 0$ and initial conditions: $y(-1) = 2, y'(-1) = 1$. The characteristic equation of this O.D.E is $(r+2)^2 = 0$, so we have two repeated real roots $r_1 = r_2 = -2$. The solution is given by $y(t) = C_1y_1 + C_2y_2$, where $y_1 = e^{-2t}$ and $y_2 = te^{-2t}$ (obtained by reduction of order). So, the general solution is:

$$y(t) = C_1e^{-2t} + C_2te^{-2t}$$

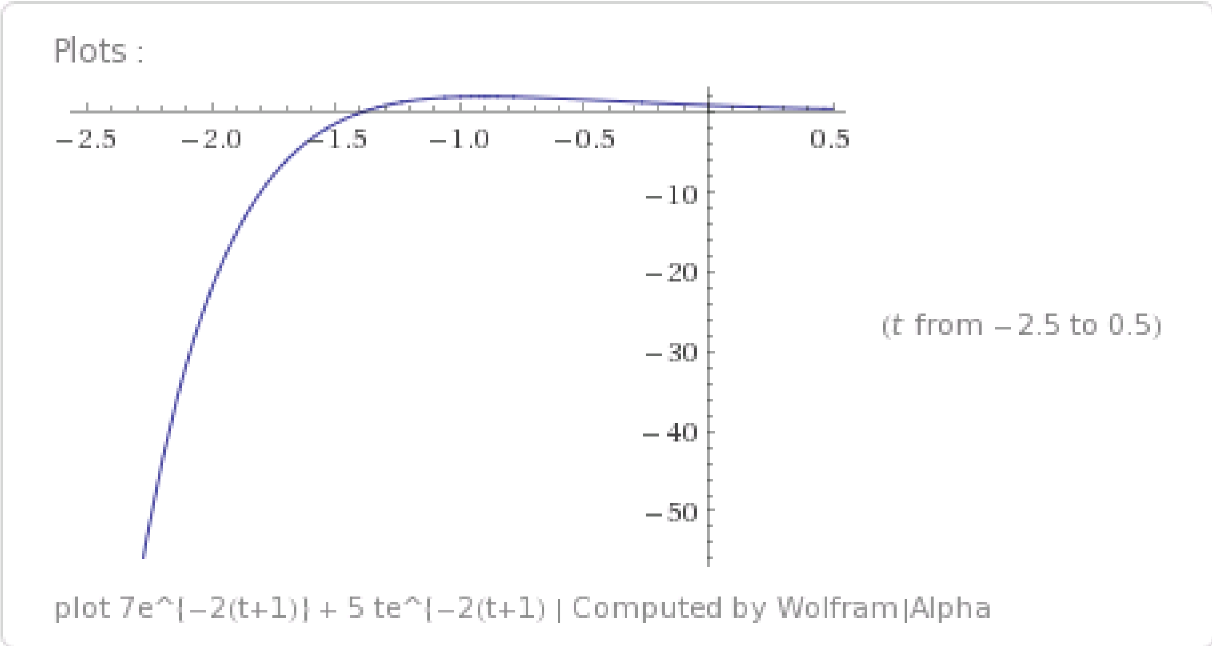
Solving for the constants C_1, C_2 using the initial conditions we obtain:

$$\begin{cases} y(-1) = 2 = c_1e^2 - C_2e^2 = e^2(C_1 - C_2) \Longrightarrow C_1 - C_2 = 2e^{-2} & \cdots (*) \\ \\ y'(-1) = 1 = -2C_1e^2 + C_2(e^2 + 2e^2) = e^2(3C_2 - 2C_1) \Longrightarrow -2C_1 + 3C_2 - e^{-2} & \cdots (**) \end{cases}$$

Multiplying $(*)$ by 2 and subtracting from $(**)$ we obtain $C_2 = 5e^{-2}$ and $C_1 = 7e^{-2}$. The solution for the the I.V.P is:

$$\boxed{y(t) = 7e^{-2(t+1)} + 5te^{-2(t+1)}}$$

The graph for this solution is:



Also, $\lim_{t\to\infty} y(t) = 0$

16. Consider the homogeneous, 2nd O.D.E with constant coefficients: $y'' - y' + \frac{1}{4}y = 0$ and initial conditions: $y(0) = 2, y'(0) = b$. The characteristic equation of this O.D.E is $(r - \frac{1}{2})^2 = 0$, so we have two repeated real roots $r_1 = r_2 = \frac{1}{2}$. The solution is given by $y(t) = C_1y_1 + C_2y_2$, where $y_1 = e^{t/2}$ and $y_2 = te^{t/2}$ (obtained by reduction of order). So, the general solution is:

$$y(t) = C_1e^{t/2} + C_2te^{t/2}$$

Solving for the constants C_1, C_2 using the initial conditions we obtain:

$$\begin{cases} y(0) = 2 = C_1 + C_2 \cdot 0 \Longrightarrow C_1 = 2 \\ \\ y'(0) = b = C_1\left(\frac{e^0}{2}\right) + C_2\left(e^0 + 0\right) = \frac{C_1}{2} + C_2 = 1 + C_2 \Longrightarrow C_2 = b - 1 \end{cases}$$

The solution for the the I.V.P is:

$$\boxed{y(t) = e^{t/2}(2 + (b-1)t)}$$

If $b - 1 \geq 0 \iff b \geq 1$, then $\lim_{t\to\infty} y(t) = \infty$.

Otherwise, if $b - 1 \leq 0 \iff b < 1$, then $\lim_{t\to\infty} y(t) = -\infty$

Therefore, the critical value for b is $\boxed{b = 1}$

25. Consider the homogeneous, 2nd O.D.E $t^2y'' + 3ty' + y = 0, \quad t > 0, \quad y_1(t) = t^{-1}$. Let us find y_2 by reduction of order: Suppose that the second solution y_2 is of the form:

$$y_2(t) = V(t) \cdot y_1(t) \Longrightarrow y_2(t) = \frac{V(t)}{t}$$

Then

$$y_2'(t) = \frac{V'(t)}{t} - \frac{V(t)}{t^2} \quad \text{and} \quad y_2''(t) = \frac{V''(t)}{t} - \frac{2V'(t)}{t^2} + \frac{2V(t)}{t^3}$$

Since y_2 is a solution, it has to satisfy the O.D.E:

$$\begin{aligned}
0 &= t^2 y_2'' + 3t y_2' + y_2 \\
&= t^2\left(\frac{V''(t)}{t} - \frac{2V'(t)}{t^2} + \frac{2V(t)}{t^3}\right) + 3t\left(\frac{V'(t)}{t} - \frac{V(t)}{t^2}\right) + \left(\frac{V(t)}{t}\right) \\
&= tV''(t) - 2V'(t) + \frac{2V(t)}{t} + 3V'(t) - \frac{3V(t)}{t} + \frac{V(t)}{t} \\
&= tV''(t) + V'(t)(3-2) + V(t)\left(\frac{2}{t} - \frac{3}{t} + \frac{1}{t}\right) \\
&= tV''(t) + V'(t)
\end{aligned}$$

Hence, we have that $tV''(t) + V'(t) = 0$. If we make the substitution: $W = V' \Longrightarrow W' = V''$, we get:

$$tW' + W = 0 \iff \frac{d}{dt}[t \cdot W] = 0 \iff t \cdot W = C \iff W = \frac{C}{t}$$

Changing the substitution boac to V:

$$W = V' = \frac{C}{t} \Longrightarrow V = Cln(t)$$

Hence, our second solution is:

$$\boxed{y_2(t) = \frac{ln(t)}{t}}$$

26. Consider the homogeneous, 2nd O.D.E $t^2 y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0, \quad y_1(t) = t$. Let us find y_2 by reduction of order: Suppose that the second solution y_2 is of the form:

$$y_2(t) = V(t) \cdot y_1(t) \Longrightarrow y_2(t) = V(t) \cdot t$$

Then

$$y_2'(t) = V'(t)t + V(t) \quad \text{and} \quad y_2''(t) = V''(t)t + 2V'(t)$$

Since y_2 is a solution, it has to satisfy the O.D.E:

$$\begin{aligned}
0 &= t^2 y_2'' - t(t+2)y_2' + (t+2)y_2 \\
&= t^2(V''t + 2V') - t(t+2)(V't + V) + (t+2)(Vt) \\
&= t^3 V'' + V'(2t^2 - t^2(t+2)) + V(-t(t+2) + t(t+2)) \\
&= t^3 V'' + V'(2t^2 - t^3 - 2t^2) \\
&= t^3 V'' - t^3 V' \\
&= V'' - V'
\end{aligned}$$

Hence, we have that $V'' - V' = 0$. If we make the substitution: $W = V' \Longrightarrow W' = V''$, we get:

$$W' - W = 0 \iff W' = W \iff W = e^t$$

Changing the substitution back to V:

$$W = V' = e^t \Longrightarrow V = e^t$$

Hence, our second solution is:

$$\boxed{y_2(t) = t \cdot e^t}$$

28. Consider the homogeneous, 2nd O.D.E $(x-1)y''-xy'+y=0 \quad x>1, \quad y_1(t)=e^x$. Let us find y_2 by reduction of order: Suppose that the second solution y_2 is of the form:

$$y_2(x)=V(x)\cdot y_1(x) \Longrightarrow y_2(x)=V(x)\cdot e^x$$

Then

$$y_2'(t)=e^x(V'+V) \quad \text{and} \quad y_2''(t)=e^x(V''+2V'+V)$$

Since y_2 is a solution, it has to satisfy the O.D.E:

$$\begin{aligned}
0 &= (x-1)y''-xy'+y \\
&= (x-1)(e^x(V''+2V'+V))-x(e^x(V'+V))+(Ve^x) \\
&= (x-1)(V''e^x+2V'e^x+Ve^x)-x(V'e^x+Ve^x)+Ve^x \\
&= xe^xV''+2xe^xV'+xe^xV-V''e^x-2V'e^x-Ve^x-xe^xV'-xVe^x+Ve^x \\
&= V''(xe^x-e^x)+V'(2xe^x-2e^x-xe^x)+V(xe^x-e^x-xe^x+e^x) \\
&= (e^x(x-1))V''+(e^x(x-2))V' \\
&= (x-1)V''+(x-2)V'
\end{aligned}$$

Hence, we have that $(x-1)V''+(x-2)V'=0$. If we make the substitution: $W=V' \Longrightarrow W'=V''$, we get:

$$(x-1)W'+(x-2)W=0 \iff \int\frac{dW}{W}=\int\frac{2-x}{x-1}dx \Longrightarrow ln(W)=ln(x-1)-x \iff W=(x-1)e^{-x}$$

Changing the substitution back to V:

$$W=V'=(x-1)e^{-x} \Longrightarrow V=e^{-x}x$$

Hence, our second solution is:

$$y_2(x)=e^{-x}x\cdot e^x \iff \boxed{y_2(x)=x}$$