M343 Homework 8

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Section 5.1

1. $\sum_{n=0}^{\infty}(x-3)^n$. To determine the radius of convergence of this series we apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(x-3)^n}\right| = |x-3|\lim_{n\to\infty}1 = |x-3|\cdot 1 < 1 \Longrightarrow -1 < x-3 < 1 \Longrightarrow 2 < x < 4$$

Hence, if $x \in (2,4)$ the series absolutely converges. By definition, the radius of convergence is $\boxed{\rho = 1}$

3. $\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$. To determine the radius of convergence of this series we apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{\frac{x^{2(n+1)}}{(n+1)!}}{\frac{x^{2n}}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}\cdot n!}{x^{2n}\cdot(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{x^2}{n+1}\right| = x^2\lim_{n\to\infty}\frac{1}{n+1} = x^2\cdot 0 = 0$$

Hence, this series converges everywhere. By definition, the radius of convergence is $\boxed{\rho = \infty}$

8. $\sum_{n=0}^{\infty}\frac{n!x^n}{n^n}$. To determine the radius of convergence of this series we apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)!x^{n+1}}{(n+1)^{n+1}}}{\frac{n!x^n}{n^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cdot x\cdot n^n}{(n+1)^{n+1}}\right| = |x|\lim_{n\to\infty}\left|\frac{n^n}{(n+1)^n}\right| = |x|\cdot\frac{1}{e} < 1 \Longrightarrow |x| < e$$

Hence, if $x \in (-e,e)$ the series absolutely converges. By definition, the radius of convergence is $\boxed{\rho = e}$

19. $\sum_{n=0}^{\infty}a_n(x-1)^{n+1}$. If we make the change $m = n+1 \Longrightarrow n = m-1$.

$$\begin{aligned}\sum_{n=0}^{\infty}a_n(x-1)^{n+1} &= \sum_{m=1}^{\infty}a_{m-1}(x-1)^m && \text{Since if } n=0 \text{ then } m=1\\ &= \sum_{n=1}^{\infty}a_{n-1}(x-1)^n && \text{Changing } m=n\text{. This shows the result.}\end{aligned}$$

20. $\sum_{k=0}^{\infty}a_{k+1}x^k + \sum_{k=0}^{\infty}a_kx^{k+1}$. If we change the power of the second series to x^k then:

$$\begin{aligned}\sum_{k=0}^{\infty}a_{k+1}x^k + \sum_{k=0}^{\infty}a_kx^{k+1} &= \sum_{k=0}^{\infty}a_{k+1}x^k + \sum_{k=1}^{\infty}a_{k-1}x^k\\ &= a_1\cdot x^0 + \sum_{k=1}^{\infty}a_{k+1}x^k + \sum_{k=1}^{\infty}a_{k-1}x^k && \text{Starting the first series from } k=1\\ &= a_1 + \sum_{k=1}^{\infty}(a_{k+1}+a_{k-1})x^k && \text{Summing the series. This shows the result.}\end{aligned}$$

21. $\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$. If we make the change $m = n-2 \Longrightarrow n = m+2$.

$$\begin{aligned}\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2} &= \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m && \text{Since if } n=2 \text{ then } m=0\\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n && \text{Changing } m=n\text{. This shows the result.}\end{aligned}$$

23. $x\sum_{n=1}^{\infty} na_nx^{n-1} + \sum_{k=0}^{\infty} a_kx^k$. Relabeling n for k in the second series:

$$\begin{aligned}
x\sum_{n=1}^{\infty} na_nx^{n-1} + \sum_{k=0}^{\infty} a_kx^k &= \sum_{n=1}^{\infty} na_nx^n + \sum_{n=0}^{\infty} a_nx^n && \text{Multiplying by } x \text{ the first sum} \\
&= \sum_{n=1}^{\infty} na_nx^n + \sum_{n=1}^{\infty} a_nx^n + a_0 && \text{Starting the second sum from 1.} \\
&= \sum_{n=1}^{\infty} (na_n + a_n)x^n + a_0 && \text{Summing the two series} \\
&= \sum_{n=1}^{\infty} (n+1)a_nx^n + a_0 && \text{Common factor } a_n \\
&= \boxed{\sum_{n=0}^{\infty} (n+1)a_nx^n} && \text{Starting from } n = 0.
\end{aligned}$$

25. $S = \sum_{m=2}^{\infty} m(m-1)a_mx^{m-2} + x\sum_{k=1}^{\infty} ka_kx^{k-1}$. Changing $k = m-2$ then $m = k+2$ in the first series:

$$\begin{aligned}
S &= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k + \sum_{k=1}^{\infty} ka_kx^k && \text{Multiplying by } x \text{ the second sum} \\
&= 2a_2 + \sum_{k=1}^{\infty} (k+2)(k+1)a_{k+2}x^k + \sum_{k=1}^{\infty} ka_kx^k && \text{Starting the first sum from 1.} \\
&= 2a_2 + \sum_{k=1}^{\infty} [(k+2)(k+1)a_{k+2} + ka_k]x^k && \text{Summing the two series} \\
&= \sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} + ka_k]x^k && \text{Starting from } k = 0 \\
&= \boxed{\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + na_n]x^n} && \text{Relabeling } k = n.
\end{aligned}$$

Section 5.2

2. Find the power series solutions of $y'' - xy' - y = 0$; about the point $x_0 = 0$.

Let $y = \sum_{n=0}^{\infty} a_nx^n$. Then, $y' = \sum_{n=1}^{\infty} na_nx^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}$. Suppose y satisfies the O.D.E.:

$$\begin{aligned}
0 &= \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} - x\left(\sum_{n=1}^{\infty} na_nx^{n-1}\right) - \sum_{n=0}^{\infty} a_nx^n \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} na_nx^n - \sum_{n=0}^{\infty} a_nx^n && \text{Changing the power of the first series} \\
&= 2a_2 - a_0 + \sum_{n=1}^{\infty} x^n[(n+2)(n+1)a_{n+2} - na_n - a_n] && \text{Starting from } n = 1 \text{ and summing the series} \\
&= \sum_{n=0}^{\infty} x^n[(n+2)(n+1)a_{n+2} - na_n - a_n] && \text{Starting from } n = 0
\end{aligned}$$

Therefore, the recurrence relation is $(n+2)(n+1)a_{n+2} - na_n - a_n = 0$. Written in standard from:

$$\boxed{a_{n+2} = \frac{a_n}{n+2}} \quad \text{for } n = 0, 1, 2, ...$$

For n even; a_0 free — For n odd; a_1 free

$$a_2 = \frac{a_0}{2} \qquad a_3 = \frac{a_1}{3}$$

$$a_4 = \frac{a_2}{4} = \frac{a_0}{2 \cdot 4} \qquad a_5 = \frac{a_1}{3} = \frac{a_3}{5} = \frac{a_1}{3 \cdot 5}$$

$$a_6 = \frac{a_4}{6} = \frac{a_0}{2 \cdot 4 \cdot 6} \qquad a_7 = \frac{a_5}{7} = \frac{a_1}{3 \cdot 5 \cdot 7}$$

$$\vdots \qquad \vdots$$

$$a_{2k} = \frac{a_0}{2^k \cdot k!} \qquad a_{2k+1} = \frac{a_1 \cdot 2^k \cdot k!}{(2k+1)!}$$

Plugging the coefficients back into the solution:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} \\ &= a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!} + a_1 \sum_{k=0}^{\infty} \frac{2^k k! x^{2k+1}}{(2k+1)!} \end{aligned}$$

Hence, $\boxed{y_1 = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}}$ and $\boxed{y_2 = \sum_{k=0}^{\infty} \frac{2^k k! x^{2k+1}}{(2k+1)!}}$. These solutions from a F.S.O.S since:

$$W(y_1, y_2)(x_0) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} (0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

The first four terms are: $y_1 = 1 + \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 4} + \frac{x^6}{1 \cdot 2 \cdot 4 \cdot 6}$ and $y_2 = x + \frac{x^3}{1 \cdot 3} + \frac{x^5}{1 \cdot 3 \cdot 5} + \frac{x^7}{1 \cdot 3 \cdot 5 \cdot 7}$

3. Find the power series solutions of $y'' - xy' - y = 0$; about the point $x_0 = 0$.

Let $y = \sum_{n=0}^{\infty} a_n (x-1)^n$. Then, $y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$. Suppose y satisfies the O.D.E.:

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - x(\sum_{n=1}^{\infty} n a_n (x-1)^{n-1}) - \sum_{n=0}^{\infty} a_n (x-1)^n \\ &= \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} (x-1)^m - ((x-1)+1)(\sum_{m=1}^{\infty} m a_m (x-1)^{m-1}) - \sum_{m=0}^{\infty} a_m (x-1)^m \\ &= \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} (x-1)^m - \sum_{m=1}^{\infty} m a_m (x-1)^m - \sum_{m=1}^{\infty} m a_m (x-1)^{m-1} - \sum_{m=0}^{\infty} a_m (x-1)^m \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} n a_n (x-1)^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n \\ &= 2a_2 - a_1 - a_0 + \sum_{n=1}^{\infty} (x-1)^n [(n+2)(n+1) a_{n+2} - n a_n - (n+1) a_{n+1} - a_n] \\ &= \sum_{n=1}^{\infty} (x-1)^n [(n+2)(n+1) a_{n+2} - (n+1) a_{n+1} - (n+1) a_n] \end{aligned}$$

Therefore, the recurrence relation is $(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} - (n+1)a_n = 0$. Written in standard from:

$$\boxed{a_{n+2} = \frac{a_{n+1} + a_n}{n+2}} \quad \text{for } n = 0, 1, 2, ...$$

For n even; a_0, a_1 free

$$a_2 = \frac{a_1 + a_0}{2}$$

$$a_4 = \frac{a_3 + a_2}{4} = \frac{3a_1 + 2a_0}{4 \cdot 3}$$

For n odd; a_0, a_1 free

$$a_3 = \frac{a_2 + a_1}{3} = \frac{3a_1 + a_0}{3 \cdot 2}$$

Plugging the coefficients back into the solution:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + \cdots \\
&= a_0 + a_1(x-1) + (\frac{a_1}{2} + \frac{a_0}{2})(x-1)^2 + (\frac{a_1}{2} + \frac{a_0}{6})(x-1)^3 + (\frac{a_1}{4} + \frac{a_0}{6})(x-1)^4 + \cdots \\
&= a_0(1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots) + a_1((x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \cdots)
\end{aligned}$$

Hence,

$$\boxed{y_1 = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots} \quad \text{and} \quad \boxed{y_2 = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \cdots}$$

These solutions from a F.S.O.S since:

$$W(y_1, y_2)(x_0) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}(1) = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$$

5. Find the power series solutions of $(1-x)y'' + y = 0$; about the point $x_0 = 0$.

Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. Suppose y satisfies the O.D.E.:

$$\begin{aligned}
0 &= (1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=1}^{\infty} (m+1)(m)a_{m+1}x^m + \sum_{m=0}^{\infty} a_m x^m \\
&= 2a_2 + a_0 \sum_{m=1}^{\infty} x^m[(m+2)(m+1)a_{m+2} - (m+1)(m)a_{m+1} + a_m]
\end{aligned}$$

Therefore, the recurrence relation is:

$$\begin{cases} 2a_2 + a_0 = 0 \quad \text{for } n = 0 \\ (n+2)(n+1)a_{n+2} - (n+1)(n)a_{n+1} + a_n = 0 \quad \text{for } n = 1, 2, \cdots \end{cases}$$

Written in standard from:

$$\begin{cases} a_2 = -\dfrac{a_0}{2} \quad \text{for } n = 0; \;\; a_0 \text{ free} \\ a_{n+2} = \dfrac{(n+1)(n)a_{n+1} - a_n}{(n+2)(n+1)} \quad \text{for } n = 1, 2, \cdots \end{cases}$$

For n even; a_0, a_1 free

$$a_2 = -\frac{a_0}{2}$$

$$a_4 = \frac{6a_3 - a_2}{4 \cdot 3} = \frac{-a_0 - 2a_1}{4 \cdot 3 \cdot 2}$$

For n odd; a_0, a_1 free

$$a_3 = \frac{2a_2 - a_1}{3 \cdot 2} = \frac{-a_0 - a_1}{3 \cdot 2}$$

Plugging the coefficients back into the solution:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ &= a_0 + a_1 x + (-\frac{a_0}{2})x^2 + (\frac{-a_0}{6} - \frac{a_1}{6})x^3 + (\frac{-a_0}{24} - \frac{a_1}{12})x^4 + \cdots \\ &= a_0(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots) + a_1(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \cdots) \end{aligned}$$

Hence, $\boxed{y_1 = 1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots}$ and $\boxed{y_2 = x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \cdots}$

These solutions from a F.S.O.S since:

$$W(y_1, y_2)(x_0) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} (0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

15. Consider the I.V.P $y'' - xy' - y = 0, \quad y(0) = 2;\ y'(0) = 1.$

The general solution was already computed in (2): $y = C_1 y_1 + C_2 y_2$, where: $\boxed{y_1 = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}}$ and $\boxed{y_2 = \sum_{k=0}^{\infty} \frac{2^k k! x^{2k+1}}{(2k+1)!}}$.

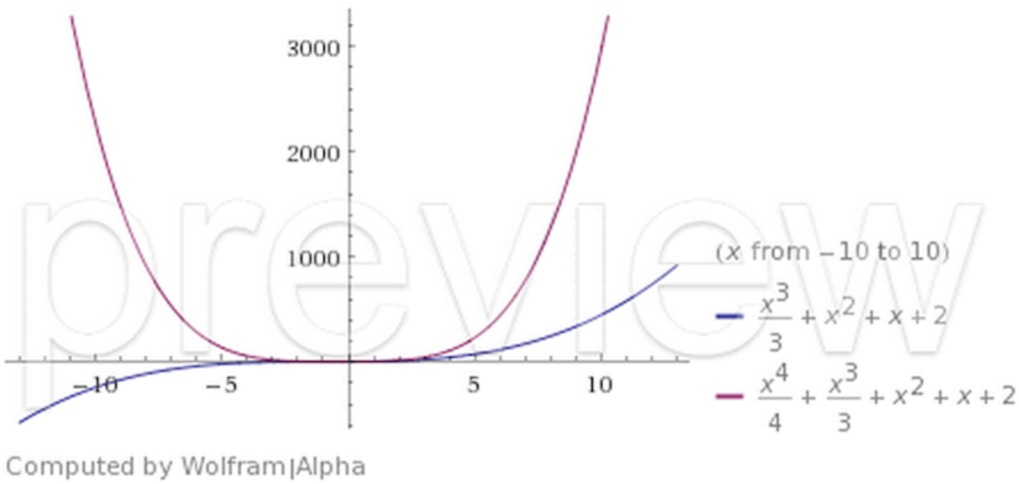
Hence,

$$\begin{cases} y(0) = 2 = C_1 + 0 \cdot C_2 \Longrightarrow \boxed{C_1 = 2} \\ y'(0) = 1 = 2(\sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!})'\Big|_{x=0} + C_2(\sum_{k=0}^{\infty} \frac{2^k k! x^{2k+1}}{(2k+1)!})'\Big|_{x=0} = 2 \cdot 0 + C_2 \Longrightarrow \boxed{C_2 = 1} \end{cases}$$

So, the solution to the I.V.P is given by

$$y = 2y_1 + y_2 \iff \boxed{y = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots}$$

From the plot below, we can see that if $|x| < 0.7$, then the four-term approximation is reasonably accurate.



17. Consider the I.V.P $y'' + xy' + 2y = 0; \quad y(0) = 4;\ y'(0) = -1$.

Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. Suppose y satisfies the O.D.E.:

$$\begin{aligned}
0 &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\Big(\sum_{n=1}^{\infty} n a_n x^{n-1}\Big) + 2\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n \quad \text{Changing the power of the first series} \\
&= 2a_2 + 2a_0 + \sum_{n=1}^{\infty} x^n[(n+2)(n+1)a_{n+2} + na_n + 2a_n] \quad \text{Starting from } n=1 \text{ and summing the series} \\
&= \sum_{n=0}^{\infty} x^n[(n+2)(n+1)a_{n+2} + na_n + 2a_n] \quad \text{Starting from } n=0
\end{aligned}$$

Therefore, the recurrence relation is $(n+2)(n+1)a_{n+2} + na_n + 2a_n$. Written in standard from:

$$\boxed{a_{n+2} = \frac{-a_n}{n+1}} \quad \text{for } n = 0, 1, 2, ...$$

For n even; a_0 free

$$a_2 = -\frac{a_0}{1}$$

$$a_4 = \frac{-a_2}{3} = \frac{a_0}{3}$$

$$a_6 = -\frac{a_4}{5} = -\frac{a_0}{3\cdot 5}$$

$\vdots$

$$a_{2k} = \frac{a_0(-1)^k}{1\cdot 3\cdot 5\cdots(2k-1)}$$

$k = 1, 2, \cdots$

For n odd; a_1 free

$$a_3 = -\frac{a_1}{2}$$

$$a_5 = -\frac{a_3}{4} = \frac{a_1}{2\cdot 4}$$

$$a_7 = -\frac{a_5}{6} = -\frac{a_1}{2\cdot 4\cdot 6}$$

$\vdots$

$$a_{2k+1} = \frac{a_1(-1)^k}{2\cdot 4\cdot 6\cdots(2k)}$$

$k = 1, 2, \cdots$

Plugging the coefficients back into the solution:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{k=0}^{\infty} a_{2k}x^{2k} + \sum_{k=0}^{\infty} a_{2k+1}x^{2k+1} \\
&= a_0\left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{1\cdot 3\cdot 5\cdots(2k-1)}x^{2k}\right) + a_1\left(x + \sum_{k=1}^{\infty} \frac{(-1)^k}{2\cdot 4\cdot 6\cdots(2k)}x^{2k+1}\right)
\end{aligned}$$

Finally, solve for a_0, a_1

$$\begin{cases} y(0) = 4 = a_0 + o\cdot a_1 \Longrightarrow \boxed{a_0 = 4} \\ y'(0) = -1 = 0\cdot a_0 + a_1 \Longrightarrow \boxed{a_1 = -1} \end{cases}$$

The solution to the I.V.P is:

$$y = 4\left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{1\cdot 3\cdot 5\cdots(2k-1)}x^{2k}\right) - \left(x + \sum_{k=1}^{\infty} \frac{(-1)^k}{2\cdot 4\cdot 6\cdots(2k)}x^{2k+1}\right) \iff$$

$$\boxed{y = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4 - \frac{1}{8}x^5 + \cdots}$$

From the plot below, we can see that if $|x| < 0.5$, then the four-term approximation is reasonably accurate.