Differential Equations Cheatsheet

Jargon

General Solution: a family of functions, has parameters.
Particular Solution: has no arbitrary parameters.
Singular Solution: cannot be obtained from the general solution.

Linear Equations

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = f(x)$$

1st-order

$$F(y', y, x) = 0 \qquad y' + a(x)y = f(x) \qquad \text{I.F.} = e^{\int a(x)dx} \qquad \textit{Sol}: y = Ce^{-\int a(x)dx}$$

Variable Separable

$$\frac{dy}{dx} = f(x, y) \qquad A(x)dx + B(y)dy = 0$$

Test:

$$f(x, y)f_{xy}(x, y) = f_x(x, y)f_y(x, y)$$

Sol: Separate and integrate on both sides.

Exact

$$M(x, y)dx + N(x, y)dy = 0 = dg(x, y)$$

$$\text{Iff} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Sol: Find $g(x, y)$ by integrating and comparing:

$$\int M dx \quad \text{and} \quad \int N dy$$

Reduction to Exact via Integrating Factor

$$I(x, y)[M(x, y)dx + N(x, y)dy] = 0$$

Case I

If $\frac{M_y - N_x}{M} \equiv h(y)$ then $I(x, y) = e^{-\int h(y)dx}$

Case II

If $\frac{N_x - M_y}{N} \equiv g(x)$ then $I(x, y) = e^{-\int g(x)dx}$

Case III

If $M = yf(xy)$ and $N = xg(xy)$ then $I(x, y) = \frac{1}{xM - yN}$

Homogeneous of degree 0

$$f(tx, ty) = t^0 f(x, y) = f(x, y)$$

Sol: Reduce to var.sep. using:

$$y = xv \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Bernoulli

$$y' + p(x)y = q(x)y^n$$

Sol: Change var $z = \frac{1}{y^{n-1}}$ and divide by $\frac{1}{y^n}$.

Reduction by Translation

$$y' = \frac{Ax + By + C}{Dx + Ey + F}$$

Case I: Lines intersect

Sol: Put $x = X + h$ and $y = Y + k$,
find h and k, solve var.sep. and translate back.

Case II: Parallel Lines $(A = B, D = E)$

Sol: Put $u = Ax + By$, $y' = \frac{u' - A}{B}$ and solve.

Principle of Superposition

If $y''+ay'+by = f_1(x)$ has solution $y_1(x)$ and $y''+ay'+by = f_2(x)$ has solution $y_2(x)$ then $y'' + ay' + by = f(x) = f_1(x) + f_2(x)$ has solution: $y(x) = y_1(x) + y_2(x)$

2nd-order Homogeneous

$$F(y'', y', y, x) = 0 \qquad y'' + a(x)y' + b(x)y = 0 \qquad \textit{Sol}: y_h = c_1 y_1(x) + c_2 y_2(x)$$

Reduction of Order - Method

If we already know y_1, put $y_2 = vy_1$,
expand in terms of v'', v', v, and put $z = v'$
and solve the reduced equation.

Wronskian (Linear Independence)

$y_1(x)$ and $y_2(x)$ are linearly independent iff

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

Constant Coefficients

$$\text{A.E.} \qquad \lambda^2 + a\lambda + b = 0$$

A. Real roots

Sol: $y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

B. Single root

Sol: $y(x) = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$

C. Complex roots

Sol: $y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$
with $\alpha = -\frac{a}{2}$ and $\beta = \frac{\sqrt{4b - a^2}}{2}$

Euler-Cauchy Equation

$$x^2 y'' + axy' + by = 0 \quad \text{where } x \neq 0$$

$$A.E.: \quad \lambda(\lambda - 1) + a\lambda + b = 0$$

Sol: $y(x)$ of the form x^λ
Reduction to Constant Coefficients: Use $x = e^t, t = \ln x$,
and rewrite in terms of t using the chain rule.

A. Real roots

Sol: $y(x) = C_1 x^{\lambda_1} + C_2 x^{\lambda_2} \qquad x \neq 0$

B. Single root

Sol: $y(x) = x^\lambda (C_1 + C_2 \ln |x|)$

C. Complex roots $(\lambda_{1,2} = \alpha \pm i\beta)$

Sol: $y(x) = x^\alpha [C_1 \cos(\beta \ln |x|) + C_2 \sin(\beta \ln |x|)]$

2nd-order Non-Homogeneous

$$F(y'', y', y, x) = 0 \qquad y'' + a(x)y' + b(x)y = f(x) \qquad \textit{Sol}: y = y_h + y_p = C_1 y_1(x) + C_2 y_2(x) + y_p(x)$$

Simple case: y', y missing

$$y'' = f(x)$$

Sol: Integrate twice.

Simple case: y missing

$$y'' = f(y', x)$$

Sol: Change of var: $p = y'$ and then solve twice.

Simple case: y', x missing

$$y'' = f(y)$$

Sol: Change of var: $p = y'$ + chain rule, then
$p\frac{dp}{dy} = f(y)$ is var.sep.
Solve it, back-replace p and solve again.

Simple case: x missing

$$y'' = f(y', y)$$

Sol: Change of var: $p = y'$ + chain rule, then
$p\frac{dp}{dy} = f(p, y)$ is 1st-order ODE.
Solve it, back-replace p and solve again.

Method of Undetermined Coefficients / "Guesswork"

Sol: Assume $y(x)$ has same form as $f(x)$ with
undetermined constant coefficients.
Valid forms:

1. $P_n(x)$
2. $P_n(x)e^{ax}$
3. $e^{ax}(P_n(x) \cos bx + Q_n(x) sinbx$

Failure case: If any term of $f(x)$ is a solution of y_h,
multiply y_p by x and repeat until it works.

Variation of Parameters (Lagrange Method)

(More general, but you need to know y_h)
Sol: $y_p = v_1 y_1 + v_2 y_2 + \cdots + v_n y_n$

$$\begin{array}{lllllll} v_1' y_1 & + & \cdots & + & v_n' y_n & = & 0 \\ v_2' y_2' & + & \cdots & + & v_n' y_n' & = & 0 \\ \cdots & + & \cdots & + & \cdots & = & 0 \\ v_n' y_b^{(n-1)} & + & \cdots & + & v_n' y_n^{(n-1)} & = & \phi(x) \end{array}$$

Solve for all v_i' and integrate.

Power Series Solutions

1. Assume $y(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, compute y', y"
2. Replace in the original D.E.
3. Isolate terms of equal powers
4. Find *recurrence relationship* between the coefs.
5. Simplify using common series expansions

(Use $y = vx$, $z = v'$ to find $y_2(x)$ if only $y_1(x)$ is known.)

Taylor Series variant

1. Differentiate both sides of the D.E. repeatedly
2. Apply initial conditions
3. Substitute into T.S.E. for $y(x)$

Validity

For $y'' + a(x)y' + b(x)y = 0$
if $a(x)$ and $b(x)$ are analytic in $|x| < R$,
the power series also converges in $|x| < R$.

Ordinary Point: Power method success guaranteed.
Singular Point: success *not* guaranteed.

Regular singular point:
if $xa(x)$ and $x^2b(x)$ have a *convergent MacLaurin series* near point $x = 0$. (Use translation if necessary.)

Irregular singular point: otherwise.

Method of Frobenius for Regular Singular pt.

$$y(x) = x^r(c_0 + c_1x + c_2x^2 + \cdots) = \sum_{n=0}^{\infty} c_n x^{r+n}$$

Indicial eqn: $r(r-1) + a_0 r + b_0 = 0$

Case I: r_1 and r_2 differ but *not by an integer*

$$\begin{aligned} y_1(x) &= |x|^{r_1} \left(\sum_{n=0}^{\infty} c_n x^n\right), \quad c_0 = 1 \\ y_2(x) &= |x|^{r_2} \left(\sum_{n=0}^{\infty} c_n^* x^n\right), \quad c_0^* = 1 \end{aligned}$$

Case II: $r_1 = r_2$

$$\begin{aligned} y_1(x) &= |x|^{r} \left(\sum_{n=0}^{\infty} c_n x^n\right), \quad c_0 = 1 \\ y_2(x) &= |x|^{r} \left(\sum_{n=1}^{\infty} c_n^* x^n\right) + y_1(x) ln|x| \end{aligned}$$

Case III: r_1 and r_2 differ by an integer

$$\begin{aligned} y_1(x) &= |x|^{r_1} \left(\sum_{n=0}^{\infty} c_n x^n\right), \quad c_0 = 1 \\ y_2(x) &= |x|^{r_2} \left(\sum_{n=0}^{\infty} c_n^* x^n\right) + c_1^* y_1(x) ln|x|, \quad c_0^* = 1 \end{aligned}$$

Laplace Transform

FIXME TODO

Fourier Transform

FIXME TODO