Differential Equations Study Sheet

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1 First Order Differential Equations

- Differential equations can be used to explain and predict new facts for about everything that changes continuously.
- $\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0$.
- t is the independent variable, x is the dependent variable, a and k are parameters.
- The order of a differential equation is the highest deriviative in the equation.
- A differential equation is linear if it is linear in parameters such that the coefficients on each deriviative of y term is a function of the independent variable (t).
- Solutions: Explicit $\rightarrow$ Written as a function of the independent variable. Implicit $\rightarrow$ Written as a function of both y and t. (defines one or more explicit solutions.

1.1 Population Model

- Model: $\frac{dP}{dt} = kP$.
- Equilibrium solution occurs when $\frac{dP}{dt} = 0$.
- Solution: $P(t) = Ae^{(kt)}$.
- If $k > 0$, then $\lim_{t\to\infty} P(t) = \infty$. If $k < 0$, then $\lim_{t\to\infty} P(t) = 0$.
- Redefine model so it doesn't blow up to infinity.
- $\frac{dP}{dt} = kP(1 - \frac{P}{N})$.
- N is the carrying capacity of the population.

1.2 Seperation of Variables Technique

- $\frac{dy}{dt} = g(t)h(y)$.
- $\frac{1}{h(y)}dy = g(t)dt$.
- Integrate both sides and solve for y.
- You might lose the solution $h(y) = 0$.

1.3 Mixing Problems

- $\frac{dQ}{dt}$ = Rate In - Rate Out.
- Consider a tank that initally contains 50 gallons of pure water. A salt solution containing 2 pounds of salt per gallon of water is poured into the tank at a rate of 3 gal/min. The solution leaves the tank also at 3 gal/min.
- Therefore Input = 2(lb/gal)*3(gal/min).
- Output = ?(lbs/gal)*3(gal/min).
- Salt in Tank = $\frac{Q(t)}{50}$.
- Therefore output of salt = $\frac{Q(t)}{50}$ (lbs/gal)*3(gal/min).
- $\frac{dQ}{dt}$ = Rate In - Rate Out = 2 lbs/gal*3gal/min - $\frac{Q(t)}{50}$ (lbs/gal)*3(gal/min).
- 6 lbs/min - $\frac{3Q}{50}$lbs/min.
- Solve via seperation of Variables.

1.4 Existance and Uniqueness

- Given $\frac{dy}{dt} = f(t, y)$. If f is continuous on some interval, then there exists at least one solution on that interval.
- If both $f(t, y)$ and $\frac{\partial}{\partial y} f(t, y)$ are continuous on some interval then an initial value problem on that interval is guaranteed to have exactly one Unique solution.

1.5 Phase Lines

- Takes all the information from a slope fields and captures it in a single vertical line.
- Draw a vertical line, label the equilibrium points, determine if the slope of y is positive or negative between each equilibrium and label up or down arrows.

1.6 Classifying Equilibria and the Linearization Theorem

- Source: solutions tend away from an equilibrium $\rightarrow f'(y_o) > 0$.
- Sink: solutions tend toward an equilibrium $\rightarrow f'(y_o) < 0$.
- Node: Nither a source or a sink $\rightarrow f'(y_o) = 0$ or DNE.

1.7 Bifurcations

- Bifurcations occur at parameters where the equilibrium profile changes.
- Draw phase lines (y) for several values of a.

1.8 Linear Differential Equations and Integrating Factors

- Properties of Linear DE: If y_p and y_h are both solutions to a differential equation, (particular and homogeneous), then $y_p + y_h$ is also a solution.
- Using the integrating factor to solve linear differential equations such that $\frac{dy}{dt} + P(t)y = f(t)$.
- The integrating factor is therefore $e^{(\int P(t)dt)}$.
- Multiply both sides by the integrating factor.
- $e^{(\int P(t)dt)}\frac{dy}{dt} + e^{(\int P(t)dt)}P(t)y = e^{(\int P(t)dt)}f(t)$.
- then via chain rule ...
- $\frac{d\{e^{(\int P(t)dt)}y\}}{dt}$ ((Integrating factor * y))= $e^{(\int P(t)dt)}f(t)$.
- Then integrate to find solution.

1.9 Integration by Parts

$\int udv = uv - \int vdu$.

2 Systems

- $\frac{dx}{dt} = ax - bxy, \frac{dy}{dt} = -cy + dxy.$
- Equilibrium occurs when both differential equations are equal to zero.
- a and c are growth effects and b and d are interaction effects.
- To verify that $x(t)$, $y(t)$ is a solution to a system, take the deriviative of each and compare them to the originial differerential equations with x and y plugged in.
- Converting a second order differential equation, $\frac{d^2y}{dt^2} = y$. Let $v = \frac{dy}{dt}$. Thus $dv = \frac{d^2y}{dt}$.

2.1 Vector Notation

- A system of the form $\frac{dx}{dt} = ax + bxy$ and $\frac{dy}{dt} = cy + exy$ can be written in vector notation.
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$$\frac{d}{dt}\mathbf{P}(t) = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} ax + bxy \\ cy + exy \end{bmatrix}. \tag{1}$$

2.2 Decoupled System

- Completely decoupled: $\frac{dx}{dt} = f(x), \frac{dy}{dt} = g(y)$.
- Partially decoupled: $\frac{dx}{dt} = f(x), \frac{dy}{dt} = g(x, y)$.

3 Systems II

- Matrix form.
- Homogeneous = $\frac{d}{dt}\mathbf{X} = \mathbf{AX}$.
- Non-homogeneous = $\frac{d}{dt}\mathbf{X} = \mathbf{AX} + \mathbf{F}$.
- Linearity Principal
- Consider $\frac{d}{dt}\mathbf{X} = \mathbf{AX}$,where

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{2}$$

- If $X_1(t)$ and $X_2(t)$ are solutions, then $k_1X_1(t) + k_2X_2(t)$ is also a solution provided $X_1(t)$ and $X_2(t)$ are linearly independent.
- Theorem: If $\mathbf{A}$ is a matrix wtih det $\mathbf{A}$ not equal to zero, then the only equilibrium piont for the system $\frac{d}{dt}\mathbf{X} = \mathbf{AX}$ is,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3}$$

3.1 Straightline Solutions, Eigencool Eigenvectors and Eigenvalues

- A straightline solution to the system $\frac{d}{dt}\mathbf{X} = \mathbf{AX}$ exists provided that,

$$\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix}. \tag{4}$$

- To determine λ, compute the det[($\mathbf{A}$ - λI)] =

$$det\begin{bmatrix} a-\lambda & b \\ c & e-\lambda \end{bmatrix} = (a-\lambda)(e-\lambda) - bc = 0. \tag{5}$$

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- This expands to the characteristic polynomial =

$$\lambda^2 - (a-d)\lambda + ae - bc = 0.$$

- Solving the characteristic polynomial provides us with the eigenvalues of A.

3.2 Stability

Consider a linear 2 dimensional system with two nonzero, real, distinct eigenvalues, λ_1 and λ_2.

- If both eigenvalues are positive then the origin is a source (unstable).
- If both eigenvalues are negative then the origin is a sink (stable).
- If the eigenvalues have different signs, then the origin is a saddle (unstable).

3.3 Complex Eigenvalues

- Euler's Formula: $e^{a+ib} = e^a e^i b = e^a cos(b) + ie^a sin(b)$.
- Given real and complex parts of a solution, the two parts can be treated as seperate independent solutions and used in the linearization theorem to determine the general solution.
- Stability: consider a linear two dimensional system with complex eigenvalues $\lambda_1 = a+ib$ and $\lambda_2 = a - ib$.
  - If a is negative then solution spiral towards the origin (spiral sink).
  - If a is positive then the solutions spiral away from the origin (spiral source).
  - If $a = 0$ the solutions are periodic closed paths (neutral centers).

3.4 Repeated Eigenvalues

- Given the system, $\frac{d}{dt}\mathbf{X} = \mathbf{AX}$ with one repeated eigenvalue, λ_1.
- If **V1** is an eigenvector, then $X_1(t) = e^{\lambda t}V_1$ is a straight line solution.
- Another solution is of the form $X_2(t) = e^{\lambda t}(tV_1 + V_2)$.
- Where $V_1 = (A - \lambda I)V_2$.
- X_1 and X_2 will be independent and the general solution is formed in the usual manner.

3.5 Zero as an Eigenvalue

- If zero is an eigenvector, nothing changes but the form of the general solution is now

$$\mathbf{X}(t) = k_1\mathrm{V}_1 + k_2e^{\lambda_2 t}\mathbf{V}_2.$$

4 Second Order Differential Equations

- Form: $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} = q(t)y = f(t)$.
- Homogeneous if $f(t) = 0$.
- given solutions y_1 and y_2 to the 2nd order differential equation, you must check the Wronskian if both solutions are from real roots of the characteristic.
- $$\mathbf{W} = det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}. \tag{6}$$
- If W is equal to 0 anywhere on the interval of consideration, then y_1 and y_2 are not linearly independent.
- General solution given y_1 and y_2 is found as usual by the linearization theorem.
- Characteristic polynomial of a 2nd order with constant coefficients: $as^2 + bs + c = 0$.
- Solutions of the form $y(t) = e^{st}$.
- $s = -\frac{b}{2a} + / - \frac{\sqrt{b^2 - 4ac}}{2a}$.
  - if $b^2 - 4ac > 0$, then two distinct real roots.
  - if $b^2 - 4ac < 0$, then complex roots.
  - $b^2 - 4ac = 0$, then repeated real roots.

4.1 Two real distinct Roots

- Two real roots, s_1 and s_2.
- General solution $= y(t) = k_1 e^{s_1 t} + k_2 e^{s-2t}$.

4.2 Complex Roots

- Complex Roots, $s_1 = p + iq$ and $s_2 = p - iq$.
- General solution $= y(t) = k_1 e^{pt} cos(qt) + k_2 e^{pt} sin(qt)$.

4.3 Repeated Roots

- Repeated Root, s_1.
- General solution $= y(t) = k_1 e^{-\frac{b}{2a}t} + k_2 t e^{-\frac{b}{a2}t}$.

4.4 Nonhomogeneous with constant coefficents

- General solution $= y(t) = y_h + y_p$.
- Polynomial $f(t)$.
  - Look for particular solution of the form $y_p = At^n + Bt^{n-1} + Ct^{n-2} + ... + Dt + E$.
- Exponential $f(t)$.
  - Look for particular solution of the form $y_p = Ae^{pt}$.
- Sine or Cosine $f(t)$.
  - Look for particular solution of the form $y_p = Asin(at) + Bcos(at)$.
- Combination $f(t)$.
  - $f(t) = P_n(t)e^{at}, \Rightarrow y_p = (At^n + Bt^{n-1} + Ct^{n-2} + ... + Dt + E)e^{at}$.
  - $f(t) = P_n(t)sin(at)$ or $P_n(t)cos(at), \Rightarrow y_p = (A1t^n + A2t^{n-1} + A3t^{n-2} + ... + A4t + A5)cos(at) + (B1t^n + B2t^{n-1} + B3t^{n-2} + ... + B4t + B5)sin(at)$.
  - $f(t) = e^{at}sin(bt)$ or $e^{at}cos(bt), \Rightarrow y_p = Ae^{at}cos(bt) + Be^{at}sin(bt)$.
  - $f(t) = P_n(t)e^{at}sin(bt)$ or $P_n(t)e^{at}cos(bt), \Rightarrow y_p = (A1t^n + A2t^{n-1} + A3t^{n-2} + ... + A4t + A5)e^{at}cos(bt) + (B1t^n + B2t^{n-1} + B3t^{n-2} + ... + B4t + B5)e^{at}sin(bt)$.
- Superposition $f(t)$.
  - If $f(t)$ is the sum of m terms of the forms previously described.
  - $y_p = y_{p1} + y_{p2} + y_{p3} + ... + y_{pm}$.

5 LaPlace Transformations

- Definition $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)dt$.
- ONLY PROVIDED THAT THE INTEGRAL CONVERGES!!! MUST BE OF EXPONENTIAL ORDER!!!
- $L\{f(t)\} = F(s)$.
- $L\{1\} = \frac{1}{s}$.
- $L\{t\} = \frac{1}{s^2}$.
- $L\{e^{at}\} = \frac{1}{s-a}$.
- $L\{sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$.
- $L\{cos(\omega t)\} = \frac{s}{s^2+\omega^2}$.
- Linear: $L\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$.

5.1 Inverse Laplace Transforms

- Linear: $L^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$.

5.2 Transform of a derivative

- $L\{f'(t)\} = sL(f(t) - f(0)$.
- $L\{f''(t)\} = s^2 L(f(t) - sf(0) - f'(0)$.