M343 Differential Equations Summary, Summer, 2013, Enrique Areyan

General Form of an nth Order Differential Equation:

$$\boxed{a_n(t,y',\cdots,y^{(n-1)})y^{(n)} + a_{n-1}(t,y',\cdots,y^{(n-2)})y^{(n-1)} + \cdots a_1(t)y' + a_0 y = g(t)}$$

Classification:

Order: the order of a differential equation is the highest derivative in the equation.

Linear.: A differential equation is linear if the coefficients on each derivative of y term is a function of **only** the independent variable, say t, i.e.:

$$\boxed{a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots a_1(t)y' + a_0 y = g(t)} \rightarrow \text{General } n\text{th order } \textbf{linear} \text{ O.D.E}$$

Solutions: Explicit $\rightarrow$ Written as a function of the independent variable: $y(t)$. Implicit $\rightarrow$ Written as a function of both y and t.

I.V.P: O.D.E comes with **initial conditions**, $y(t_0) = y_0$, if it is a 1st O.D.E and $y(t_0) = y_0, y'(t_0) = y'_0$ for 2nd Order

First Order Differential Equations:

General Form: $\boxed{y' = f(t,y)}$. To apply methods and theorems use $\boxed{y'' + p(t)y' = g(t)}$.

Existence and Uniqueness, **linear** 1st O.D.E: Consider the I.V.P: $y' + p(t)y = g(t), \quad y(t_0) = y_0$.
IF $p(t)$ and $g(t)$ are continuos on (α,β) AND $t_0 \in (\alpha,\beta)$ **THEN** there is a unique solution to the I.V.P in (α,β).

Existence and Uniqueness, 1st O.D.E: Consider the I.V.P: $y' = f(t,y), \quad y(t_0) = y_0$.
IF f and $\frac{\partial f}{\partial y}$ are continuos on $(\alpha,\beta) \times (\gamma,\delta)$ AND $(t_0,y_0) \in (\alpha,\beta) \times (\gamma,\delta)$ **THEN** there is a unique solution to the I.V.P in some "small box" $t_0 - h < t < t_0 + h$ that is contained in $\alpha < t < \beta$. Note: here the value of y_0 matter.

Note: if the hypothesis are not meet, that does not mean that there is no solution, there may be none or many!

Types of 1st O.D.E: Separable - Linear (Integrating factor) - Bernoulli (change $u = y^{1-n}$) - Exact.

Separable: $\boxed{y' = g(t)h(y)} \iff \dfrac{dy}{dt} = g(t)h(y) \iff \dfrac{dy}{h(y)} = g(t)dt$. Integrate to solve. Might lose the solution $h(y) = 0$.

Linear: (1) Convert to standard form $y' + p(t)y = g(t)$, (2) use integrating factor $\boxed{\mu(t) = e^{\int p(t)dt}}$, (3) cross multiply $\mu(t)[y' + p(t)y = g(t)]$, (4) product rule $\dfrac{d}{dt}[\mu(t)y] = \mu(t)g(t)$, (5) integrate $\mu(t)y = \int(\mu(t)g(t)dt)$, solve right hand side, do not forget constant, and then divide by $\mu(t)$.

Bernoulli: $\boxed{y' + p(t)y = g(t)y^n}$. If $n = 0, 1$ then Bernoulli equations are just linear equations. Otherwise, make the change $u = y^{n-1} \implies u' = (1-n)y^{-n}y'$ to obtain a linear equation solvable by integrating factor. Once solved for u change back the solution to y.

Exact: $\boxed{M(x,y) + N(x,y)y' = 0}$ is exact $\iff \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, since second derivatives of continuos functions are the same. Theorem: $M(x,y) + N(x,y)y' = 0$ is exact $\iff \exists$ a unique function $\psi(x,y)$ such that $\frac{\partial \psi}{\partial x} = M; \frac{\partial \psi}{\partial y} = N$. Then: $M + Ny' = 0 \iff \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}y' = 0 \iff \frac{d}{dx}\psi(x,y) = 0 \iff \psi(x,y) = C$.
Types of problems: (1) check if exact. It it is: $\psi(x,y) = \int \frac{\partial \psi}{\partial x}\partial x = \int M \partial x = f(x,y) + h(y)$, where $h(y)$ is a pure function of y. Find $h(y)$ from: $\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}[f(x,y) + h(y)] = N'(x,y) + h'(y) = N(x,y)$, hence from $h'(y)$ we integrate $h(y) = \int h'(y)dy$. The solution if $\boxed{\psi(x,y) = f(x,y) + h(y)}$. Note that we could have use $\psi(x,y) = \int \frac{\partial \psi}{\partial y}\partial y$. Finally, try to write y in explicit form. (2) if it is not exact but $\mu(x,y)$ is given, cross multiply and then solve as in (1). Finally, (3) it is not exact. Use $\mu(x)$ or $\mu(y)$ by solving $\frac{u'}{u} = \frac{\partial M/\partial y - \partial N/\partial x}{N}$ or $\frac{u'}{u} = \frac{\partial N/\partial x - \partial M/\partial y}{M}$, cross multiply and test if it is exact. If it is, solve as in (1). (Note: $\mu(x,y)$ is not unique, an easier μ makes the problem easier).

Euler's method Approximate the solution of $y' = f(t,y), \quad y(t_0) = y_0$. Then, $y_n = y_{n-1} + hf(t_{n-1}, y_{n-1})$, $t_n = t_0 + nh$. h is the step size. This is a one step numerical method $O(n)$. In short: $\boxed{y(t_0 + nh) = y(t_n) = y_n = y_{n-1} + hf(t_{n-1}, y_{n-1})}$

Modeling with 1st O.D.E: Tank problems: $Q(t)$ = quantity of "salt" at time t. $\frac{dQ}{dt}$ = rate salt in - rate salt out. $q(t) = \frac{Q(t)}{V(t)}$ is the concentration of "salt" at time t. If $r_1 = r_2$, then $V(t)$ is constant. Else, solve $\frac{dV}{dt} = r_1 - r_2$ to get $V(t)$. Model: $\boxed{\dfrac{dQ}{dt} = r_1 \cdot q_0 - r_2 \cdot q(t)}$. **Always check units!**. This is a 1st O.D.E solvable by integrating factor or separating variables.

Second Order Linear Differential Equations:

General Form: $\boxed{y' = f(t, y, y')}$. To apply methods and theorems use $\boxed{y'' + p(t)y' + q(t)y = g(t)}$.

Existence and Uniqueness, **linear** 2nd O.D.E: Consider the I.V.P: $y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, y'(t_0) = y_1$.
IF $p(t), q(t), g(t)$ are continuos on (α, β) AND $t_0 \in (\alpha, \beta)$ **THEN** there is a unique solution to the I.V.P in (α, β).

Superposition Principle: Let $L(y) := y'' + p(t)y' + q(t)y = 0 \cdots (*)$ (homogeneous). **IF** y_1, y_2 are two solutions of $(*)$ **THEN** $y = C_1y_1 + C_2y_2$ is also a solution of $(*)$.

Definition: the Wronskian of two functions f, g is $W(f, g)(t) = (f \cdot g' - f' \cdot g)$ (Wronskian is the det(f,g——f',g'))

Theorem **IF** y_1, y_2 are two solutions of the O.D.E (only! **no** I.V.P) $y'' + p(t)y' + q(t)y = 0$ AND the initial condition $y(t_0) = y_0; y'(t_0) = y_0'$ is assigned, **THEN** is is always possible to choose constants C_1, C_2 s.t. $y = C_1y_1 + C_2y_2$ is a solution of the I.V.P $\iff$ the Wronskian is not zero at t_0

Theorem **IF** y_1, y_2 are two solutions of $y'' + p(t)y' + q(t)y = 0 \cdots (*)$, **THEN** the family of solutions $y = C_1y_1 + C_2y_2$, include every solution of $(*)$ $\iff$ $\exists t_0$ s.t. $W(y_1, y_2)(t_0) \neq 0$

Definition: **IF** y_1, y_2 are two solutions of $y'' + p(t)y' + q(t)y = 0 \cdots (*)$ **THEN** $\{y_1, y_2\}$ form a fundamental set of solutions (F.S.O.S) of $(*)$ $\iff$ $\exists t_o \in \mathbb{R}$ s.t. $W(y_1, y_2)(t_0) \neq 0$

Theorem: Consider $y'' + p(t)y' + q(t)y = 0 \cdots (*)$ whose coefficients $p(t), q(t)$ are continuous on I. Choose $t_0 \in I$. Let y_1 be the solution of $(*)$ that satisfies $y_1(t_0) = 1; y_1'(t_0) = 0$. Let y_2 be the solution of $(*)$ that satisfies $y_2(t_0) = 0; y_2'(t_0) = 1$. **THEN** $\{y_1, y_2\}$ is a F.S.O.S since $W(y_1, y_2)(t_0) = 1$.

2nd O.D.E. with constant coefficients: $\boxed{ay'' + by' + cy = g(t)}$. If $g(t) \equiv 0$, then this is called the homogenous equation. Otherwise it is the non-homogeneous. For the homogeneous case: Suppose the solution is $y(t) = e^{rt}$. Then, $y'(t) = re^{rt}$ and $y''(t) = r^2e^{rt}$. Plug in the equation: $e^{rt}(ar^2+br+c) = 0$, since e^{rt} is never zero, we get the characteristic equation: $\boxed{ar^2 + br + c = 0}$. The solution depends on the root of this equation. However, in any case, by superposition principle, the general solution is $y(t) = C_1y_1 + C_2y_2$, where y_1, y_2 are the homogenous solutions.

Different real roots: Roots r_1, r_2. General solution $y = C_1e^{r_1t} + C_2e^{r_2t}$. Note that these solutions form a F.S.O.S since $W(e^{r_1t}, e^{r_2t}) = (r_2 - r_1)e^{(r_1+r_2)t} \neq 0$ for any t.

Complex roots: $r_{1,2} = \lambda \pm i\mu$. (Euler's formula $e^{i\theta} = cos(\theta) + isin(\theta)$). The F.S.O.S is given by $\{e^{\lambda t}cos(\mu t), e^{\lambda t}sin(\mu t)\}$, since $W(e^{\lambda t}cos(\mu t), e^{\lambda t}sin(\mu t)) = \mu e^{2\lambda t} \neq 0$ for any t. (we assume $\mu \neq 0$, otherwise we are in the repeated roots case).

Repeated real roots: $r_1 = r_2 = r$. The F.S.O.S is given by $\{e^{rt}, te^{rt}\}$, since $W(e^{rt}, te^{rt}) = e^{2rt} \neq 0$ for any t.

Reduction of order: Given y_1 a solution of $y'' + p(t)y' + q(t)y = 0 \cdots (*)$, assume $y_2 = V(t)y_1$. Find $V(t)$ knowing that y_2 satisfies $(*)$. You end up with an equation like $p(t)V'' + q(t)V' = 0$. Substitute $W = V' \implies W' = V''$, solve for W as a 1st O.D.E (linear), and then change back to V. Finally, obtain a second solution $\boxed{Y_2 = V(t)y_1}$

Method of Undetermined Coefficients $ay'' + by' + cy = g(t) \neq 0$, where $g(t) = e^{at}$ OR $g(t) = cos(bt); sin(bt)$ OR $g(t) = P_n(t)$ OR a combination of these. Then the solution is given by: $\boxed{y_{general} = y_h + y_p}$, where y_h is L.I from y_p.
y_h is the solution to the homogenous equation $ay'' + by' + cy = 0$.
y_p is the particular solution guess at from $g(t)$ with undetermined coefficients. We will consider only the following 3 cases:

exp. $g(t) = e^{at}$, guess $y_p = Ae^{at}$

trig. $g(t) = cos(bt)$ or $g(t) = sin(bt)$, guess $y_p = Acos(bt) + Bsin(bt)$

poly. $g(t) = P_n(t) = a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0$, guess $y_p = A_nt^n + A_{n-1}t^{n-1} + \cdots + A_1t + A_0$

Or combinations of these, in which case y_p = sum of guess, each of functions of the above form.

Method of Variations of Parameters: $y'' + p(t)y' + q(t)y = g(t) \neq 0$, where $g(t)$ is any function. Need standard form.

The solution is $\boxed{y = u_1y_1 + u_2y_2}$, where y_1, y_2 are a F.S.O.S, .i.e. $y_h = C_1y_2 + C_2y_2$. Find u_1, u_2 with

$$\boxed{u_1' = \frac{W_1 \cdot g}{W(y_1, y_2)} = \frac{-y_2 \cdot g}{W}} \quad \text{and} \quad \boxed{u_2' = \frac{W_2 \cdot g}{W(y_1, y_2)} = \frac{y_1 \cdot g}{W}}$$

Higher Order Linear Differential Equations:

Existence and Uniqueness, **linear** Higher O.D.E: Consider the I.V.P:

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0 y = g(t), \quad y(t_0) = y_0, y'(t_0) = y_0', \cdots, y^{(n-1)}(t_0) = y_0^{(n-1)}$$

IF $t_0 \in I$ is such that $p_{n-1}, \cdots, p_0$ and $g(t)$ are continuous on I **THEN** there exists y a solution of the I.V.P on I.

Given $y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0 y = 0$, there are n- linearly independent solutions so that $\{y_1, y_2, \cdots, y_n\}$ form a F.S.O.S. on I if there exists at least one point $t_0 \in I$ such that $W(y_1, y_2, \cdots, y_n)(t_0) \neq 0$. The homogeneous solution is then $\boxed{y_h = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n}$.

Higher O.D.E. with constant coefficients:
$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y = 0$. Solve the characteristic equation: $a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 = 0$. Solution will depend on the roots just as in the case for the 2nd O.D.E. Make sure that the solutions are linearly independent, piece by piece.

Method of Undetermined Coefficients: exactly the same as 2nd O.D.E, but in higher dimensions.

Method of Variations of Parameters: $a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y = g(t)$, where $g(t)$ is any function. Suppose sol.:

$$\boxed{y = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n} \quad \text{where, } y_1, y_2, \cdots, y_n \text{ are the homogeneous solutions}$$

In general: $\boxed{u_i' = \dfrac{W_i \cdot g(t)}{W}}$, where $W = W(y_1, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_{n-1}^{(n-1)} \end{vmatrix} \quad W_i = \begin{vmatrix} y_1 & \cdots & 0 & \cdots & y_n \\ y_1' & \cdots & 0 & \cdots & y_n' \\ \vdots & \vdots & \vdots & & \\ y_1^{(n-1)} & \cdots & 1 & \cdots & y_{n-1}^{(n-1)} \end{vmatrix}$,

in the ith column. To obtain u_i, simply integrate.

Power Series Solution of Linear 2nd O.D.E: $P(x)y'' + Q(x)y' + R(x)y = g(x)$ (not necessarily constant coefficients).
First, review of power series: $\sum_{n=0}^{\infty} a_n(x - x_0)^n$, (power series about x_0).

Power series is convergent at the point x_1 if $\sum_{n=0}^{\infty} a_n(x_1 - x_0)^n < \infty$. P.S. divergent at the point x_1 if $\sum_{n=0}^{\infty} a_n(x_1 - x_0)^n = \pm\infty$.
Power series are functions whose **domain is only** those point where it converges. We exclude the points where it diverges. A power series can be convergent everywhere or it may converge for only some values of x.

The power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is called absolutely convergent at x_1 if: $\sum_{n=0}^{\infty} |a_n(x_1 - x_0)^n| = \sum_{n=0}^{\infty} |a_n||(x_1 - x_0)^n| < \infty$.

Theorem: **IF** $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is absolutely convergent at x_1 **THEN** it is convergent at x_1. (not necessarily the other way).

Ratio Test: Consider the series $\sum_{n=0}^{\infty} b_n$, where b_n is a number. Then:

1. **IF** $\lim_{n\to\infty} \left|\dfrac{b_{n+1}}{b_n}\right| < 1$ **THEN** the series is absolutely convergent.
2. **IF** $\lim_{n\to\infty} \left|\dfrac{b_{n+1}}{b_n}\right| > 1$ **THEN** the series is divergent.
3. **IF** $\lim_{n\to\infty} \left|\dfrac{b_{n+1}}{b_n}\right| = 1$ **THEN** the test is inconclusive.

We apply this test for power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ as follow: $\lim_{n\to\infty} \left|\dfrac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\dfrac{a_{n+1}(x - x_0)^{n+1}}{a_n(x - x_0)^n}\right| = |x - x_0| \lim_{n\to\infty} \left|\dfrac{a_n + 1}{a_n}\right|$

In general, to find the points of convergence, apply the ratio test to get absolute convergence when the series is < 1 and divergence when > 1, and test the boundaries $= 1$ separately. To test boundaries remember:

1. Theorem: **IF** $\lim_{n\to\infty} b_n \neq 0$ **THEN** the series diverges.
2. p-test: $\sum_{n=0}^{\infty} \dfrac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$ (justification using integrals).
3. Alternating series: $\sum_{n=0}^{\infty} (-1)^n b_n$ is convergent if $\lim_{n\to\infty} b_n = 0$; is divergent if $\lim_{n\to\infty} b_n \neq 0$

Taylor series: given $f(x)$: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ is the taylor expansion of $f(x)$. To be able to write the Taylor Series, f should have n-th derivative. Only analytic functions (very nice functions - all derivatives exists at least in a small neigh. of x_0) have taylor series.

Shifting index: shifting index in series is very important to apply methods to solve 2nd O.D.E. Remember that to sum power series the powers of $(x-x_0)$ must agree and the indices must agree. First make the power $(x-x_0)^n$ agree and then the beginning of the series, possibly taking out terms.

Power Series Solution of Linear 2nd O.D.E: $P(x)y'' + Q(x)y' + R(x)y = g(x)$; case $g(x) = 0$ homogeneous case. Idea: find the power series that represents the solution of the O.D.E.

Definition: Consider $P(x)y'' + Q(x)y' + R(x)y = 0$. We say that x_0 is an ordinary point of the O.D.E if $P(x_0) \neq 0$. Otherwise, $P(x_0) = 0$, x_0 is a singular point of the O.D.E. (The point is to choose a center x_0 for the power series solution such that the point is "nice", i.e., an ordinary point).

Assuming that the solution can be represented as a power series about $x_0 \in \mathbb{R}; P(x_0) \neq 0$, an ordinary point, then $y = \sum_{n=0}^{\infty} a_n(x-x_0)^n$, $y' = \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1}$, $y = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}$, plug in the O.D.E and find coeff. a_n.
After plugging back into the equation, arrange series so that they can be summed and obtain the recurrence relation.
From recurrence relation try to find a general form of coefficients, or just find the first four or five. Plug back into the solution for y to obtain two solutions y_1, y_2.

Theorem: Consider $P(x)y'' + Q(x)y' + R(x)y = 0 \cdots (*)$ a linear, 2nd O.D.E. Let x_0 be an ordinary point $P(x_0) \neq 0$. Write in standard form:

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0; \quad \text{Let } p(x) = \frac{Q(x)}{P(x)} \text{ and } q(x) = \frac{R(x)}{P(x)}$$

IF $p(x)$ AND $q(x)$ are analytic at x_0 **THEN:**

(1) The solution of $(*)$ can be written as: $y = a_0y_1 + a_1y_2$, where y_0 and y_1 are analytic at x_0.

(2) Moreover, $\{y_1, y_2\}$ form a F.S.O.S

(3) Let ρ_1 be the radius of convergence of $p(x)$ and ρ_2 be the radius of convergence of $q(x)$.
Then, the radius of convergence of y, call it ρ, is such that $\rho \geq min\{\rho_1, \rho_2\}$

Note: in this course we'll use the definition of a function being analytic when all its derivatives exists and are continuous in a small ball around x_0. Equivalently, a function is analytic if the power series about x_0 can be used to express the function. (Analytic is more restrictive than continuous, which was the only condition for E.U.T. in the first part of the course).

Euler's Equations: $P(x)y'' + Q(x)y' + R(x)y = 0$. Consider x_0 to be a singular point, i.e., $P(x_0) = 0$.
We call Euler Equation:

$$\boxed{(x-x_0)^2y'' + \alpha(x-x_0)y' + \beta y = 0}$$

We want to seek a solution about x_0. Note that all linear; 2nd O.D.E with singular points can be reduced to euler's equation. Assume the solution is $y = (x-x_0)^r$, for $r \in \mathbb{R}$. The idea is to find r. So y satisfies the equation. Also, $y' = r(x-x_0)^{r-1}$ and $y'' = r(r-1)(x-x_0)^{r-2}$. Plug and play. Since $(x-x_0)^r$ is never zero around x_0, we can cancel it to obtain the characteristic equation:

$$\boxed{r^2 + (\alpha-1)r + \beta = 0}$$

As before, find the two roots: r_1, r_2. The solution will be given according to these roots:

Different real roots: Roots r_1, r_2. General solution $y = C_1(x-x_0)^{r_1} + C_2(x-x_0)^{r_2}$.

Repeated real roots: $r_1 = r_2 = r$. General solution $y = C_1(x-x_0)^r + C_2(x-x_0)^r ln(|x-x_0|)$

Complex roots: $r_{1,2} = \lambda \pm i\mu$. General solution $y = (x-x_0)^{\lambda}[C_1 cos(\mu ln(|x-x_0|) + C_2 sin(\mu ln(|x-x_0|))]$

Note that for Euler's equation $P(x) = (x-x_0)^2 = 0 \iff x = x_0$, has a solution everywhere but when $x = x_0$. Since the solution we assume to be a power function $y = x^r$, $x \neq x_0$, then the domain of the function is either to the left or right, i.e., $x > x_0$ or $x < x_0$, depending on the initial values given.