Practice Problems for the final

(1) Determine the general solution of the given O.D.E.

$$y^{(4)} - y = \frac{3}{t}$$

$\underline{y_h}$: $\quad y^{(4)} - y = 0$. Characteristic Equation:

$$r^4 - 1 = 0$$

we know one root is $r_1 = 1$. Hence;

$$\begin{array}{l|l} r^4 - 1 & r-1 \\ \underline{-r^4 + r^3} & r^3 + r^2 + r + 1 \\ r^3 - 1 & \\ \underline{-r^3 + r^2} & \\ r^2 - 1 & \\ \underline{-r^2 + r} & \\ r - 1 & \\ \underline{-r + 1} & \\ 0 & \end{array}$$

$\Rightarrow \quad r^4 - 1 = (r-1)(r^3 + r^2 + r + 1)$

We also know that $r_2 = -1$ is another root. Hence,

$$\begin{array}{l|l} r^3 + r^2 + r + 1 & r+1 \\ \underline{-r^3 - r^2} & r^2 + 1 \\ r + 1 & \\ \underline{-r - 1} & \\ 0 & \end{array}$$

$\Rightarrow \quad r^3 + r^2 + r + 1 = (r+1)(r^2+1)$
$= (r+1)(r+i)(r-i)$.

Hence, $r^4 - 1 = (r-1)(r+1)(r+i)(r-i)$.

So the homogeneous solution is:

$y_h = C_1 e^t + C_2 e^{-t} + C_3 \cos(t) + C_4 \sin(t) + \cancel{C_5 \cos(-t) + C_6 \sin(-t)}$

$y_h = C_1 e^t + C_2 e^{-t} + C_3 \cos(t) + C_4 \sin(t) + \cancel{C_5 \cos(t) + C_7 \sin(t)}$; $\cancel{C_7 = C_6}$

therefore, the homogeneous solution is:

$$\boxed{y_h = C_1 e^t + C_2 e^{-t} + C_3 \cos(t) + C_4 \sin(t) + \cancel{C_5 t \cos(t) + C_6 t \sin(t)}}$$

the general solution is given by:

$$\boxed{y_g = y_h + y_p}$$

By Variation of parameters:

let $y_1 = e^{-t}$; $y_2 = e^{+t}$; $y_3 = \cos(t)$; $y_4 = \sin(t)$

the general solution is:

$$y_g = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4 \text{ ; where}$$

$U_i = \dfrac{W_i \cdot g}{W}$, where : $W(y_1, y_2, y_3, y_4) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$

$$= \begin{vmatrix} e^{-t} & e^{t} & \cos(t) & \sin(t) \\ -e^{-t} & e^{t} & -\sin(t) & \cos(t) \\ e^{-t} & e^{t} & -\cos(t) & -\sin(t) \\ -e^{-t} & e^{t} & \sin(t) & -\cos(t) \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & e^{t} & \cos(t) & \sin(t) \\ 0 & e^{t} & -\sin(t) & \cos(t) \\ 0 & e^{t} & -\cos(t) & -\sin(t) \\ 1 & e^{t} & \sin(t) & -\cos(t) \end{vmatrix} \quad ; \quad W_3 = \begin{vmatrix} e^{-t} & e^{t} & 0 & \sin(t) \\ -e^{-t} & e^{t} & 0 & \cos(t) \\ e^{-t} & e^{t} & 0 & -\sin(t) \\ -e^{-t} & e^{t} & 1 & -\cos(t) \end{vmatrix}$$

(1) $y''' - y' = 3\sin t.$ ②

By undetermined coefficients:

since this equation can be thought of as a 2^{nd} o.D.E $u = y'$ $\rightarrow$ $u'' - u = 3\sin(t)$

$$y_g = y_h + y_p.$$

y_h: $y''' - y' = 0$; CHARACTERISTIC EQUATION: $r^3 - r = 0 \Leftrightarrow r(r^2-1) = 0$

$\Leftrightarrow r(r-1)(r+1) = 0 \Rightarrow r_1 = 0;\ r_2 = 1;\ r_3 = -1.$

$$\boxed{y_h = C_1 + C_2 e^{t} + C_3 e^{-t}}$$

y_p: $y_p = A\cos(t) + B\sin(t)$. y_p satisfies the equation:

$y_p' = -A\sin(t) + B\cos(t)$. $y_p'' = -A\cos(t) - B\sin(t)$

$y_p''' = A\sin(t) - B\cos(t).$

$$y_p''' - y_p' = 3\sin t.$$

$$A\sin(t) - B\cos(t) + A\sin(t) - B\cos(t) = 3\sin t$$

$$2A\sin(t) - 2B\cos(t) = 3\sin t$$

$$\begin{cases} 2A = 3 \Rightarrow \boxed{A = 3/2} \\ -2B = 0 \Rightarrow \boxed{B = 0} \end{cases}$$

$$\boxed{y_p = \frac{3}{2}\cos(t)}$$

So, the general solution is:

$$\boxed{y = C_1 + C_2 e^{t} + C_3 e^{-t} + \frac{3}{2}\cos(t)}$$

We can check the sol $y' = C_2 e^{t} - C_3 e^{-t} - \frac{3}{2}\sin(t)$; $y'' = C_2 e^{t} + C_3 e^{-t} - \frac{3}{2}\cos(t)$.

$y''' = C_2 e^{t} - C_3 e^{-t} + \frac{3}{2}\sin(t)$

$y''' - y' =$

$\cancel{C_2 e^{t}} - \cancel{C_3 e^{-t}} + \frac{3}{2}\sin(t) - \cancel{C_2 e^{t}} + \cancel{C_3 e^{-t}} + \frac{3}{2}\sin(t) = \sin(t)\left(2 \cdot \frac{3}{2}\right) = \boxed{3\sin(t)}$

Solution WORKS!

(2) $4x^2y'' + 8xy' + 17y = 0$; $y(1) = 2$, $y'(1) = -3$. ③

$\Leftrightarrow \quad x^2y'' + 2xy' + \frac{17}{4}y = 0$

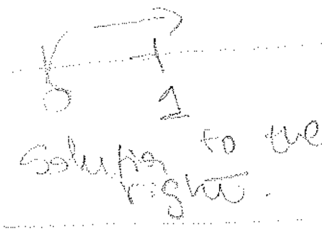
$(x-x_0)^2y'' + \alpha(x-x_0)y' + \beta y = 0$

this is an eulers equation — x_0 a singular point.

the characteristic equation is: $r^2 + (\alpha - 1)r + \beta = 0$

In our case: $P(x) = x^2$; so $x_0 = 0$ is a singular point.

the equation can be written as:



$$(x-0)^2y'' + 2(x-0)y' + \frac{17}{4}y = 0.$$

So, the solution is given by solving:

$$r^2 + (2-1)r + \frac{17}{4} = 0 \quad \Leftrightarrow \quad r^2 - r + \frac{17}{4} = 0$$

$$r = \frac{1 \pm \sqrt{1-17}}{2 \cdot 1} = \frac{1 \pm \sqrt{-16}}{2} = \frac{1 \pm 4i}{2} = \boxed{\frac{1}{2} \pm 2i}.$$

this exercise would work as a power series solution b/c $P(x=1) \neq 0$, ordinary

CASE: two complex roots.

the solution is:

$$y = (x-0)^\lambda\left[c_1 \cos(\mu \ln(|x-x_0|)) + c_2 \sin(\mu \ln(|x-x_0|))\right]$$

$$y = x^{1/2}\left[c_1 \cos(2\ln(|x|)) + c_2 \sin(2\ln(|x|))\right]$$

Solving for c_1, c_2:

$$y(1) = \boxed{2 = c_1}$$

$$y'(1) = -3 = \frac{1}{2}c_1 + \left[-c_1 \sin(2\ln(0)) \cdot [\;] + \frac{c_2 \cos(0)}{|x|}\right]$$

$$= \frac{1}{2}c_1 + c_2 = 1 + c_2 \Rightarrow \boxed{c_2 = -4}$$

. the solution to the I.V.P is

$$\boxed{y = x^{1/2}\left[2\cos(2\ln(|x|)) - 4\sin(2\ln(|x|))\right]}$$

the solution oscillates as $x \to \infty$

(3) $(4-x^2)y''+2y=0$; $x_0=0$. (4)

$$y=\sum_{n=0}^{\infty}a_n x^n \;;\; y'=\sum_{n=1}^{\infty}n a_n x^{n-1} \;;\; y''=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}$$

$$(4-x^2)\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}+2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}4n(n-1)a_n x^{n-2}-\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$m=n-2$; $n=m+2$; $m=n$; $m=n$

$$\sum_{m=0}^{\infty}4(m+2)(m+1)a_{m+2}x^m-\sum_{m=2}^{\infty}m(m-1)a_m x^m+\sum_{m=0}^{\infty}2a_m x^m=0$$

$$8a_2+24a_3x+2a_0+2a_1x+\sum_{m=2}^{\infty}\left[4(m+2)(m+1)a_{m+2}-m(m-1)a_m+2a_m\right]x^m=0$$

$$\begin{cases} x(24a_3+2a_1)+(8a_2+2a_0)=0 \\ 4(m+2)(m+1)a_{m+2}+a_m(-m(m-1)+2)=0 \end{cases}$$

$$\begin{cases} 24a_3+2a_1=0 \Rightarrow a_3=\frac{-2}{24}a_1 \Rightarrow \boxed{a_3=\frac{-1}{12}a_1} \\ 8a_2+2a_0=0 \Rightarrow a_2=-\frac{2}{8}a_0 \Rightarrow \boxed{a_2=-\frac{1}{4}a_0} \\ a_{m+2}=\dfrac{a_m(m(m-1)-2)}{4(m+2)(m+1)} \;;\; m=2,3,4,\ldots \end{cases}$$

a_1, a_0 free

$\underline{m=2}$:

$a_4=0$

$\underline{m=4}$:

$a_6=0$

$\vdots$

$a_{2k}=0$; $k>1$.

$\underline{m=3}$:

$$a_5=\frac{4a_3}{80}=\frac{a_3}{20}=\frac{\frac{-1}{12}a_1}{20}=\boxed{-\frac{a_1}{240}=a_5}$$

$\underline{m=5}$:

$$a_7=\frac{a_5 18}{165}=\frac{6a_5}{55}=\frac{6\left(-\frac{a_1}{240}\right)}{55}=\frac{-\frac{a_1}{40}}{55}=\boxed{\frac{-a_1}{2200}=a_7}$$

the solution is :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \dots$$

$$y = a_0 + a_1 x \left(-\frac{1}{4} a_0\right) x^2 \left(-\frac{1}{12} a_1\right) x^3 + \left(0 \cdot x^4\right) + \left(-\frac{a_1}{240} x^5\right) + \left(0 \cdot x^6\right) + \left(-\frac{a_1}{2200} x^7\right) + \dots$$

$$\boxed{y = a_0\left(1 - \frac{1}{4}x^2\right) + a_1\left(x - \frac{1}{12}x^3 - \frac{1}{240}x^5 - \frac{1}{2200}x^7 + \dots\right)}$$

Hence,

$$\boxed{y_1 = 1 - \frac{1}{4}x^2}$$

$$\boxed{y_2 = x - \frac{1}{12}x^3 - \frac{1}{240}x^5 - \frac{1}{2200}x^7 + \dots}$$

We can check that y_1 is a sol.

$$y_1 = 1 - \frac{1}{4}x^2 \; ; \; y_1' = -\frac{1}{2}x \; ; \; y_1'' = -\frac{1}{2}$$

$$(4 - x^2) y_1'' + 2y_1 = (4 - x^2)\left(-\frac{1}{2}\right) + 2\left(1 - \frac{1}{4}x^2\right)$$

$$= -2 + \frac{x^2}{2} + 2 - \frac{x^2}{2} = 0 \Rightarrow \boxed{y_1 \text{ is a sol!}}$$

Factoring:

$$a_{m+2} = \frac{m-2}{4(m+2)} a_m$$

$$a_{2k+1} = -\frac{1}{4^k} \frac{(2k-3)!}{(2k+1)!} a_1$$

(3) $y''-xy'-y=0$; $x_0=1$. (5)

$$y=\sum_{n=0}^{\infty} a_n(x-1)^n \ ; \ y'=\sum_{n=1}^{\infty} n a_n(x-1)^{n-1} \ ; \ y''=\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - x\sum_{n=1}^{\infty} n a_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - ((x-1)+1)\sum_{n=1}^{\infty} n a_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} - \sum_{n=1}^{\infty} n a_n(x-1)^{n} - \sum_{n=1}^{\infty} n a_n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$m=n-2$, $n=m+2$; $m=n$; $m=n-1$, $n=m+1$; $m=n$

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}(x-1)^m - \sum_{m=1}^{\infty} m a_m(x-1)^m - \sum_{m=0}^{\infty}(m+1)a_{m+1}(x-1)^m - \sum_{m=0}^{\infty} a_m(x-1)^m = 0$$

$$2a_2-a_1-a_0+\sum_{m=1}^{\infty}(x-1)^m\left[(m+2)(m+1)a_{m+2}-ma_m-(m+1)a_{m+1}-a_m\right]=0$$

$$\sum_{m=0}^{\infty}(x-1)^m\left[(m+2)(m+1)a_{m+2}-(m+1)a_{m+1}-(m+1)a_m\right]=0$$

<u>the recurrence relation is:</u>

$(m+2)(m+1)a_{m+2}-(m+1)a_{m+1}-(m+1)a_m=0$; $m=0,1,2,\dots$

$$\Rightarrow \boxed{a_{m+2}=\frac{a_{m+1}+a_m}{(m+2)}}$$

<u>$m=0$:</u>

$\boxed{a_2=\frac{a_1+a_0}{2}}$, a_1, a_0 free

<u>$m=1$:</u>

$$a_3=\frac{a_2+a_1}{3}=\frac{\left(\frac{a_1+a_0}{2}\right)+a_0}{3}=\frac{\frac{a_1+3a_0}{2}}{3}$$

$$\Rightarrow \boxed{a_3=\frac{a_1+3a_0}{2\cdot 3}}$$

<u>$m=2$:</u>

$$a_4=\frac{a_3+a_2}{4}$$

$$=\frac{\frac{a_1+3a_0}{2\cdot 3}+\frac{a_1+a_0}{2}}{4}=\frac{\frac{a_1+3a_0+3a_1+3a_0}{2\cdot 3}}{4}=\boxed{\frac{6a_0+4a_1}{2\cdot 3\cdot 4}=a_4}$$

the solution is :

$$y = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + \cdots$$

$$y = a_0 + a_1(x-1) + \left(\frac{a_1}{2} + \frac{a_0}{2}\right)(x-1)^2 + \left(\frac{a_1}{6} + \frac{a_0}{2}\right)(x-1)^3 + \left(\frac{a_0}{4} + \frac{a_1}{6}\right)(x-1)^4 + \cdots$$

$$\boxed{y = a_0\left(1 + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \cdots\right) + a_1\left((x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots\right)}$$

Hence, $\boxed{y_1 = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \cdots}$

$$\boxed{y_2 = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots}$$

(4) $y''' - 3y'' + 2y' = t + e^t$. $(t+0) + e^t$ ⑥

$y_g = y_h + y_p$ $(t+1)$

$\underline{y_h}$: $y''' - 3y'' + 2y' = 0$. CHARACTERISTIC EQUATION:

$r^3 - 3r^2 + 2r = 0 \iff r(r^2 - 3r + 2) = 0$

$r(r-2)(r-1) = 0 \Rightarrow r_1 = 0\ ;\ r_2 = 2\ ;\ r_3 = 1$

$\boxed{y_h = C_1 + C_2 e^{2t} + C_3 e^t}$

$\underline{y_p}$: $y_p = t(At + B) + (Ce^t)t$. Hence;

$\boxed{y_p = At^2 + Bt + Cte^t}$

$y'' - y' = 2e^t \sin(t)$

$y_g = y_h + y_p$

$\underline{y_h}$: $y'' - y' = 0$. CHARACTERISTIC EQUATION:

$r^2 - r = 0 \iff r(r-1) = 0;\ r_1 = 0,\ r_2 = 1$

$\boxed{y_h = C_1 + C_2 e^t}$

$\underline{y_p}$: $\boxed{y_p = (A\cos(t) + B\sin(t))e^t}$

$A_0 + (B_0 t + B_1)e^t$

(5) Determine the radius of convergence of the power series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}.$$

Using the ratio test:

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(n+1)^2(x+2)^{n+1}}{3^{n+1}}}{\frac{(-1)^n n^2 (x+2)^n}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)^2(x+2)^{n+1} \cdot 3^n}{(-1)^n n^2 (x+2)^n \cdot 3^{n+1}} \right|$$

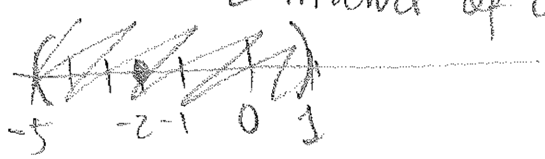
$$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{(x+2)}{-3} \right| = \frac{|x+2|}{3} \cdot \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \right|$$

$$= \frac{|x+2|}{3} \lim_{n\to\infty} \left| \frac{n^2+2n+1}{n^2} \right| = \frac{|x+2|}{3} \cdot 1 < 1 \Rightarrow |x+2| < 3$$

$\Rightarrow$ the radius of convergence is $\boxed{\rho = 3}$

$$|x+2| < 3 \Rightarrow -3 < x+2 < 3 \Rightarrow -5 < x < 1$$

interval of absolute convergence.



(6) $2y'' - 3y' + y = t - 1$, $y(0) = y'(0) = 1$.

$\underline{y_h}$: $2y'' - 3y' + y = 0$. CHARACTERISTIC EQUATION:

$$2r^2 - 3r + 1 = 0$$

$$r = \frac{3 \pm \sqrt{9-8}}{2 \cdot 2} = \frac{3 \pm 1}{4} \Rightarrow \boxed{r_1 = 1} ; \boxed{r_2 = \frac{1}{2}}$$

CASE: two different real roots:

$$\boxed{y_h = c_1 e^{t} + c_2 e^{t/2}}$$

Let $y_1 = e^t$; $y_2 = e^{t/2}$ then, the solution is given by:

$$y = U_1 y_1 + U_2 y_2 \; ; \text{ where}$$

$U_i' = \frac{W_i \cdot g}{W}$, where, $g(t) = t - 1$;

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & e^{t/2} \\ e^t & \frac{e^{t/2}}{2} \end{vmatrix} = \frac{e^{\frac{3}{2}t}}{2} - e^{\frac{3}{2}t}$$

$$= e^{\frac{3}{2}t}\left(\frac{1}{2} - 1\right) = \boxed{-\frac{1}{2} e^{\frac{3}{2}t}}$$

$$W_1 = \begin{vmatrix} 0 & e^{t/2} \\ 1 & \frac{e^{t/2}}{2} \end{vmatrix} = -e^{t/2} \; ; \quad W_2 = \begin{vmatrix} e^t & 0 \\ e^t & 1 \end{vmatrix} = e^t .$$

$$u_1' = \frac{-e^{t/2}(t-1)}{-\frac{1}{2} e^{\frac{3}{2}t}} \qquad u_2' = \frac{e^t(t-1)}{-\frac{1}{2} e^{\frac{3}{2}t}} = -u_1'.$$

$$u_1' = \frac{2\left[e^{t/2} t - e^{t/2}\right]}{e^{3/2 t}} \Rightarrow u_1' = \frac{2e^{t/2} t}{e^{3/2 t}} - \frac{2e^{t/2}}{e^{3/2 t}}$$

$$u_1' = \frac{2t}{e^t} - \frac{2}{e^t}$$

$u_1' = \frac{2t}{e^t} - \frac{2}{e^t} \Rightarrow u_1 = \int \frac{2t}{e^t} - \frac{2}{e^t} = 2\int \frac{t}{e^t}\,dt - 2\int \frac{1}{e^t}\,dt$

$= 2\int t e^{-t} - 2\int e^{-t} = 2\int t e^{-t} + 2e^{-t}$. We need only to compute:

$\int t e^{-t}\,dt$. <u>By parts</u>: $u = t \quad du = dt$
$v = e^{-t} \quad dv = -e^{-t}$

$\int u\,dv = uv - \int v\,du \Rightarrow \int t(-e^{-t}) = te^{-t} - \int e^{-t}\,dt$

$-\int te^{-t} = te^{-t} + e^{-t}$

$\Rightarrow \int te^{-t} = -te^{-t} - e^{-t}$.

$u_1 = -2te^{-t} - 2e^{-t} + 2e^{-t} \Rightarrow \boxed{u_1 = -2te^{-t} + C_1}$

<u>CHECK</u>: $u_1' = -2e^{-t} + 2te^{-t} = e^{-t}(2t-2) = \frac{2t-2}{e^t} = \frac{2t}{e^t} - \frac{2}{e^t}$ ✓

$u_2' = -u_1' \Rightarrow u_2 = -u_1 \Rightarrow \boxed{u_2 = 2te^{-t} + C_2}$

<u>So, the solution is</u>:

$$y = (-2te^{-t} + c_1)e^t + (2te^{-t} + c_2)(e^{t/2})$$

$$y = C_1 e^t + C_2 e^{t/2} - 2t + 2te^{-t/2}; \quad \boxed{y = C_1 e^t + C_2 e^{t/2} + 2t(e^{-t/2} - 1)}$$

<u>Find C_1, C_2</u>:

$$y(0) = 1 = C_1 + C_2$$

$$y'(0) = 1 = C_1 + \tfrac{1}{2}C_2 + 2 - 2 = C_1 + \tfrac{1}{2}C_2$$

$$\begin{cases} 1 = C_1 + C_2 \\ 1 = C_1 + \tfrac{1}{2}C_2 \end{cases}$$

$0 = \frac{1}{2}C_2 \Rightarrow \boxed{C_2 = 0} \Rightarrow \boxed{C_1 = 1}$

<u>So, the solution to the I.V.P is</u>:

$$\boxed{y = e^t + 2t(e^{-t/2} - 1)}$$

(8) $y'' + 4y' + 6xy = 0$ For either $x_0 = 0$ or $x_0 = 4$.

$P(x) = 4$; analityc everywhere;

AND $q(x) = 6x$; $\Rightarrow \rho \geq \min\{\rho_1, \rho_2\} \Rightarrow \boxed{\rho = \infty}$

$(1-x^2)y'' + 4xy' + y = 0$; write in standard form:

$$y'' + \frac{4x}{1-x^2}y' + \frac{1}{1-x^2}y = 0$$

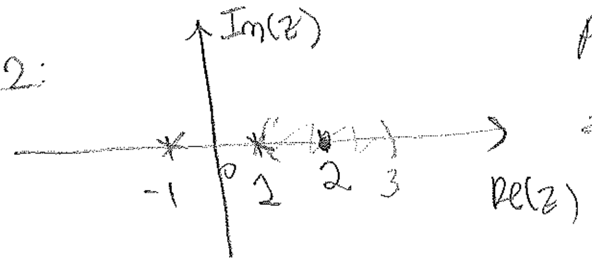
Let $p(x) = \dfrac{4x}{1-x^2}$, $q(x) = \dfrac{1}{1-x^2}$

these functions are analitiy everywhere except when:

$$1 - x^2 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

So, since $x_0 = 2$ is an ordinary point $(P(x_0) = P(2) = (1-2^2) \neq 0)$ the lower bound of radius of convergence of the solution is given by:

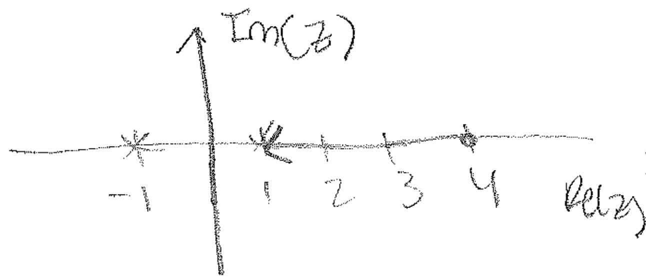
For $x_0 = 2$:



$\rho_1 = \rho_2 = 1$

$\Rightarrow \boxed{\rho \geq 1}$

For $x_0 = 4$: Note that this is also an ordinary point.



$\rho_1 = \rho_2 = 3$

$\Rightarrow \boxed{\rho \geq 3}$

(7) First check that x_0 is ordinary.

$$(x^2-2x) = (x-1)^2 - 1$$

Recursive formula:

$$a_{m+2} = \frac{-(m+1)a_{m+1} - (m^2-1)a_m}{-(m+2)(m+1)}, \quad m=2,3,\cdots$$

$$\boxed{a_{m+2} = \frac{a_{m+1} + (m-1)a_m}{m+2}}, \quad m=2,3,\cdots$$