Homework - MS403

Due Thursday, November 14, 2013

Remember to write on only one side of the sheet.

1. Let $C \in M_n(F)$ and suppose $v^t C w = v^t w$ for all $v, w \in F^n$. Prove that $C = I_n$.

2. For any field F we can define the orthogonal group $O_n(F)$ as follows: $O_n(F) = \{A \in M_n(F) | A^t A = I_n\}$. It easy to see that this is a group, a subgroup of $GL_n(F)$.

(a) Prove that $O_2(F) = \{\begin{pmatrix} a & b \\ \pm b & \mp a \end{pmatrix} | a, b \in F \text{ and } a^2 + b^2 = 1\}$.

(b) Find $|O_2(\mathbf{F}_7)|$ and $|O_2(\mathbf{F}_{11})|$. Identify the group $O_2(\mathbf{F}_7)$.

3. Determine whether the following matrix over $\mathbf{F}_{11}$ can be diagonalized:

$$\begin{pmatrix} 5 & 5 & -1 \\ -2 & 4 & -5 \\ 2 & -3 & 6 \end{pmatrix}$$

4. (a) Let A be an $m \times m$ matrix over F and let B be an $n \times n$ matrix over F. Show that if C is any $m \times n$ matrix over F then the following holds:

$$det \begin{pmatrix} A & C \\ 0_{n \times m} & B \end{pmatrix} = det(A) det(B)$$

(b) Let V be an F-vector space and let $T : V \to V$ be a linear transformation. Let W be a T-invariant subspace. Let $\overline{T} : V/W \to V/W$ be the induced linear transformation. Prove that $ch_{T_{|W}}(x) ch_{\overline{T}}(x) = ch_T(x)$.

5. Prove that if B and C are orthonormal bases for $\mathbf{R}^n$, then the change of basis matrix from B to C is orthogonal.

6. Let W be a subspace of $\mathbf{R}^n$ and let $w_1, w_2, \ldots, w_k$ be a basis for W. Define vectors $\tilde{w}_1, \tilde{w}_2, \ldots \tilde{w}_k$ as follows:

$$\tilde{w}_1 = w_1$$

and for $0 < i < k$,

$$\tilde{w}_{i+1} = w_{i+1} - \sum_{j=1}^{i} \frac{(w_{i+1}, \tilde{w}_j)}{(\tilde{w}_j, \tilde{w}_j)} \tilde{w}_j$$

Prove that $\tilde{w}_1, \tilde{w}_2, \ldots \tilde{w}_k$ is an orthogonal basis for W.

7. (a) Prove that if n is an integer, $n \geq 3$, then D_n, the dihedral group of order $2n$, is isomorphic to a subgroup of $O_2(\mathbf{R})$.

(b) Prove that $O_2(\mathbf{R})$ is the group of symmetries of the unit circle, that is the group of all one-to-one and onto functions from the unit circle to itself that preserve distance.