Homework - S403

Due October 1, 2013

Please write on only one side of the sheet.

1. Consider the following two matrices in $GL_2(\mathbf{C})$:

$$x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Let $z = xy$.

(a) Show that the set $Q_8 = \{\pm 1, \pm x, \pm y, \pm xy\}$ is a subgroup of $GL_2(\mathbf{C})$ and write out its group table.

(b) Find all the subgroups of Q_8 and prove that every subgroup is normal.

(c) Find $Z(Q_8)$ and identify the group $Q_8/Z(Q_8)$.

2. Let G be a group and let N be a normal subgroup. Let $\pi : G \to G/N$ denote the canonical homomorphism. Recall that we have shown that if H is any subgroup of G then HN is also a subgroup. Prove that if H is a subgroup of G then $\pi(H) = \pi(HN)$. Then prove that if H and K are subgroups of G, then $\pi(H) = \pi(K)$ if and only if $HN = KN$.

3. (a) Let G be a group and let x, y be distinct elements in G of order 2. Prove that if x and y commute then $\{e, x, y, xy\}$ is a subgroup of G isomorphic to $C_2 \times C_2$.

(b) Let G be a finite abelian group of order 8. Prove that G is isomorphic to one of the following 3 groups: $C_8, C_4 \times C_2$, or $C_2 \times C_2 \times C_2$.

4. (a) Let N be a normal subgroup of a group G. Prove that the one-to-one correspondence π between the subgroups of G that contain N and all of the subgroups of G/N preserves normal subgroups, that is, if K is a subgroup of G containing N, then K is normal in G if and only if $\pi(K)$ is normal in G/N.

(b) Prove that every finite group G has a homomorphic image that is a simple group, that is, a nontrivial group with no normal subgroups other than $\{e\}$ and itself.

5. From the book, problem 11.8 on page 74.

6. From the book, problem 11.9 on page 74.

7. From the book, problem 12.2 on page 74.