Homework - MS403

Due Tuesday, October 8, 2013

Remember to write on only one side of the sheet.

1. Consider the group $(\mathbf{Q}, +)/(\mathbf{Z}, +)$, the group of rationals (under addition) modulo the subgroup of integers. So an element of this group is a coset $a + \mathbf{Z}$ where a is a rational number.

(a) Find the order of the elemnt $3/4 + \mathbf{Z}$.

(b) Show that every element of this group has finite order.

(c) Prove that the group is infinite.

(d) Prove that every finite subgroup is cyclic.

2. (a) Find all possible cycle structures for elements of S_5.

(b) Find all possible orders for elements of S_5.

(c) Find the number of elements in each conjugacy class in S_5.

(d) For each conjugacy class choose a representative of that class and describe its centralizer (In each case it is a group you know or a product of groups you know).

3. Let G be a group and $Aut(G)$ denote the group of automorphisms of G, that is, the group of isomorphisms $f : G \to G$. (Convince yourself that this is indeed a group under composition.) Recall that in class we defined, for each $x \in G$, an automorphism I_x given by $I_x(g) = xgx^{-1}$ for all $g \in G$. This is called the <u>inner</u> automorphism determined by x. Let $Inn(G) = \{I_x | x \in G\}$.

(a) Prove that if $x \in G$ and $\sigma \in Aut(G)$, then $\sigma I_x \sigma^{-1} = I_{\sigma(x)}$.

(b) Prove that $Inn(G)$ is a normal subgroup of $Aut(G)$.

(c) Define a map $\alpha : G \to Inn(G)$ by $\alpha(x) = I_x$. Prove that α is a homomorphism and determine its kernel.

(d) Recall that $Z(g)$ denotes the center of G. Prove that the quotient group $G/Z(G)$ is isomorphic to $Inn(G)$.

4. (a) Prove that S_n is generated by $(1, 2)$ and $(1, 2, 3, \ldots, n)$.

(b) Let $1 \leq i < j \leq n$. Find necessary and sufficient conditions on i, j so that (i, j) and $(1, 2, 3, \ldots, n)$ generate S_n.

5. (a) Prove that $Gl_3(\mathbf{R})$ is isomorphic to $R^{\times} \times Sl_3(\mathbf{R})$.

(b) This is not true if you replace 3 by 2. What's the explanation? - no proof required.