Homework - MS403

Due Tuesday, October 15, 2013

Remember to write on only one side of the sheet.

1. Let G be a group. Prove that if $G/Z(G)$ is cyclic, then G is abelian.

2. Let m and n be positive integers. Prove that $C_m \times C_n$ is cyclic if and only if m and n are relatively prime.

3. Let G be a finite group. The exponent of G is the smallest positive integer k such that for all $g \in G$, $g^k = e$. It is denoted $exp(G)$. Prove the following:

(a) $exp(G) = lcm\{o(g)|g \in G\}$

(b) $exp(G)$ divides $|G|$.

(c) Compute the exponents of the following groups: C_6, S_4, Q_8.

4. Let G be a finite abelian group. Prove that G is cyclic if and only if $exp(G) = |G|$.

5. (a) Let V and W be vector spaces over a field F and let $T : V \to W$ be a linear transformation. Prove that if T is an isomorphism (that is, T is one-to-one and onto), then the inverse function T^{-1} is also a linear transformation.

(b) Now let $A \in M_n(F)$ and let $L_A : F^n \to F^n$ denote the linear transformation given by left multiplication by A. Prove that L_A is an isomorphism if and only if the matrix A is invertible.

6. Let V be an F-vector space and let W be a subspace.

(a) Prove that V is finite dimensional if and only if W and V/W are finite dimensional.

(b) Now assume V is finite dimensional and prove that $dim(W) + dim(V/W) = dim(V)$.

7. Let V and U be vector spaces over a field F and let $T : V \to U$ be a linear transformation.

(a) Prove that $T(V)(= \{T(v)|v \in V\})$ is a subspace of W and is finite dimensional if V is finite dimensional.

(b) Prove that if V is finite dimensional then $dim(ker(T)) + dim(T(V)) = dim(V)$ (Hint: Problem 6 and the fundamental theorem).