Homework - MS403

Due Tuesday, November 5, 2013

Remember to write on only one side of the sheet.

1. Let V be a finite dimensional F-vector space. A linear transformation $T : V \to V$ is called idempotent if $T^2 = T$. Prove that if T is an idempotent linear transformation then there is a basis B of V such that the matrix of T with respect to B has the following form:

$$\begin{pmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & 0_{m\times m} \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix and $0_{r\times s}$ denotes the $r \times s$ zero matrix.

2. Let V be a finite dimensional F-vector space. A linear transformation $T : V \to V$ is called nilpotent if $T^k = T$ for some positive integer k.

(a) Prove that if T is a nilpotent linear transformation then there is a vector $v \neq 0$ in V such that $T(v) = 0$.

(b) Prove that if W is a T-invariant subspace of V then both $T_{|W}$ and the induced linear transformation $\overline{T}$ on V/W are nilpotent.

(c) Prove that if T is a nilpotent linear transformation then there is a basis B of V such that the matrix of T with respect to B is strictly upper triangular (that is, all of the entries on the diagonal or below are zero).

3. Let $A = (a_{i,j}) \in M_n(F)$ where F is a field. Define the trace of A to be $\sum_{i=1}^{n} a_{i,i}$, the sum of the diagonal elements of A. We will denote it $Tr(A)$.

(a) Prove that the function $Tr : M_n(F) \to F$ given by sending A to $Tr(A)$ is a linear transformation.

(b) Prove that for all $A, B \in M_n(F)$, $Tr(AB) = Tr(BA)$.

(c) Let $S : V \to V$ be a linear transformation and let B, C be bases of V. Prove that $Tr(m_B(S)) = Tr(m_C(S))$. Give a definition of the trace of a linear transformation.

4. From the book, page 126, problem 2.3

5. From the book, page 126, problem 3.4

6. From the book, page 128, problem 6.4