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(1). (a) (i) $*$ is associative.

(ii) $\exists e \in G$ s.t. $\forall g \in G$: $e * g = g * e = g$

(iii) $\forall g \in G$: $\exists g^{-1} \in G$: $g * g^{-1} = g^{-1} * g = e$

(b) there exists $g \in G$ s.t $\langle g \rangle = G$; where $\langle g \rangle$ is the subgroup generated by g.

(c) $C_G(g) = \{x \in G \mid gx = xg\}$. All elements of G that commute with g.

(d) (i) $\forall v, w \in V$: $T(v+w) = T(v) + T(w)$

(ii) $\forall \alpha \in F : \forall v \in V$: $T(\alpha v) = \alpha T(v)$

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(2) (a). Any abelian group will do, for example $(\mathbb{Z}_2, \oplus)$ ✗

✗ (b) In $\frac{(\mathbb{Q},+)}{(\mathbb{Z},+)}$; the subgroup $\langle \frac{3}{4} + \mathbb{Z} \rangle = H$, $|H| < \infty$; then $[G:H]$ is finite but $\frac{(\mathbb{Q},+)}{(\mathbb{Z},+)}$

(c) $C_{15} \times C_3$; (cyclic of order 15 $\times$ cyclic of order 3; not cyclic since gcd(15

(d) the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is s.t $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$: order 2 inside of $Gl_2(\mathbb{R})$. matrices are not commutative in general

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(3) Fundamental theorem of homomorphism for groups. Let G_1, G_2 be groups and $f: G_1 \to G_2$ be a homomorphism. Let $N = \ker(f)$. then, $\frac{G_1}{N} \cong f(G_1)$. Moreover, the following diagram commutes:

$$\begin{tikzcd} G_1 \arrow[r, "f"] \arrow[d, "\pi"'] & G_2 \\ G_1/N \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

, where $\pi: G_1 \to G_1/N$, given by $\pi(g) = gN$ is the cannonical homomorphism, and $\varphi \circ \pi = f$; the homomorphisms are uni

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Note that if f is onto, then $\frac{G_1}{N} \cong G_2$.

(4) (a). the exponent is the lcm of the different possible orders of elements in S_6. these possible different orders can be computed by looking at the lenght of different cycle decompositions of elements.

$(123)(456) \to \theta = 3$

$(12)(3456) \to \theta = 4$

$(12)(345)(6) \to \theta = 6$

$(12)(34)(56) \to \theta = 2$

$(123456) \to \theta = 6$

$(1)(2)(34)(56) \to \theta = 2$.

$(12345)(6) \to \theta = 5$

$(1234)(5)(6) \to \theta = 4$. -1

Hence, the exponent is $\boxed{lcm(1,2,3,4,5,6)}$ =?

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(6) By the formula: $|G| = |C_x||C_G(x)|$; since we know $|G| = |S_7| = 7!$;
it suffices to compute $|C_x|$ to get $|C_G(x)|$; where $x = (135)(246)(7) \in S_7$
Since $C_x = \{ \sigma x \sigma^{-1} | \sigma \in S_7 \}$; and conjugation does not change cycle structure, we can count the number of permutations with two-three cycles and one-1 cycle to count $|C_x|$. these are: $\frac{7 \times \cancel{6}^{2} \times 5}{\cancel{3}} \cdot \frac{4 \times \cancel{3} \times 2}{\cancel{3}} \cdot \frac{1}{\cancel{2}} = 7 \times 5 \times 4 \times 2 = 35 \times 4 \times 2 = 70 \times 4 = \boxed{280}$ -1

$\Rightarrow |C_x| = 280$.

therefore: $|C_G(x)| = \frac{|S_7|}{|C_x|} = \frac{7!}{280} = \frac{\cancel{7} \times \cancel{6}^{6} \times \cancel{5} \times \cancel{4} \times 3 \times \cancel{2} \times 1}{\cancel{7} \times \cancel{5} \times \cancel{4} \times \cancel{2}} = \boxed{6 \cdot 3} \cancel{\ \ } = 18$ ✓ 8/10

(5) Suppose there exists a homomorphism $f: D_4 \to C_4$. By definition, for every $g_1, g_2 \in D_4$: $f(g_1 g_2) = f(g_1) f(g_2)$. Moreover, inverses are preserved under homomorphism, as well as the identity. But then: 3/10 ?

$$R_1 \cdot R_3 = I \Rightarrow f(R_1 R_3) = f(I) = e$$
$$f(R_1) \cdot f(R_3) = e \Rightarrow f(R_1) = f(R_3)^{-1}$$

So now: $f(R_1) \cdot f(R_3)^{-1} = f(R_1) f(R_3^{-1}) = f(R_1 \cdot R_3^{-1}) = f(R_1 \cdot R_1) = f(R_2)$

$f(R_3) \cdot f(R_3)^{-1} = e = f(R_3) \cdot f(R_1) = f(R_2) \Rightarrow f(R_2) = e$; a contradiction.

There is no such homomorphism.

(6) (a) True.
(b) true.
(c) false.
(d) true ✓ 12

(7) $(\mathbb{Z}_{24}, \oplus)$ has a generator call it g s.t $\langle g \rangle = \mathbb{Z}_{24}$. Every power of g that is relatively prime with 24 and is between 1 and 24 is also a generator for the whole group. Possibilities are: 5, 7, 11, 13, 17, 19 and 23.
that is: $\langle g \rangle = \langle g^5 \rangle = \langle g^7 \rangle = \langle g^{11} \rangle = \langle g^{13} \rangle = \langle g^{17} \rangle = \langle g^{19} \rangle = \langle g^{23} \rangle = \mathbb{Z}_{24}$

(8) Let $\dim(V) < \infty$, $\dim(W) < \infty$. $S: V \to W$, $T: W \to V$.
(a) Suppose $\dim(V) > \dim(W)$. W.T.S $T \circ S: V \to V$ is not an isomorphism.
By theorem proved in class, since V is finite dimensional and $T \circ S$ goes from V to itself, it suffices to show that $T \circ S$ is not 1-1 to show that it is not isomorphic

Equivalently, it suffices to show that $Ker(T \circ S) \neq \{e\}$; and then $T \circ S$ would not be 1-1; hence not an isomorphism. By hypothesis we have that $dim(V) > dim(W)$. Moreover, by the dimension theorem:

$$dim(V) = dim(Ker(S)) + dim(Im(S)).$$

Since $S: V \to W$; $dim(V) > dim(W) \Rightarrow dim(Ker(S)) > 0 \Rightarrow$
$dim(Im(S)) < dim(V)$.

Suppose for a contradiction that $Ker(T \circ S) = \{e\}$. then

$$dim(V) = dim(Ker(T \circ S)) + dim(Im(T \circ S)) = dim(Im(T \circ S))$$

But $dim(Im(S)) < dim(V) \Rightarrow dim(Im(T \circ S)) < dim(V)$ a contradiction. therefore, $Ker(T \circ S) \neq \{e\}$, $\Rightarrow$ $T \circ S$ is not an isomorphism.

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(b) $A \in M_{m \times n}(F)$; $B \in M_{n \times m}(F)$. Suppose $m > n$. We proved in class that for every linear transformation L_A there is a matrix associated with it. Hence, $L_A = A : F^m \to F^n$; $L_B = B : F^n \to F^m$. Hence, the question of $AB \neq I$; reduces to the question of $L_A \circ L_B \neq Id$; that is L_A has no right inverse (conversely, L_B has no left inverse). Note that $L_A \circ L_B : F^n \to F^n$ is also a L.T. (we proved that composition of LT is a LT in class). therefore; by part (a); we can conclude that $L_A \circ L_B$ is not an isomorphism; hence L_A and L_B are not invertible, and in the framework of matrices $L_A = A$; $L_B = B \Rightarrow AB \neq I$.

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