

$H \trianglelefteq G$ :  
 Def:  $\forall g \in G, \forall h \in H: ghg^{-1} \in H$ .  
 $\forall g \in G: gHg^{-1} = H$   
 $\forall g \in G: gH = Hg$

$$Z(G) = \{g \in G \mid gx = xg \ \forall x \in G\}$$

$$C_x = \{g \in G \mid gxg^{-1} = x\}$$

$$C(x) = \{g \in G \mid gx = xg\}$$

**Midterm-MS403**

$N_G(g) = \{g \in G \mid gs = s^{-1}g\}$   
 $gN = N^{-1}g \Rightarrow \varphi(g, N) = \varphi(g, N^{-1})$

(20)1. Complete the following definitions:

- (a) Let  $G$  be a group and  $g \in G$ . The order of  $g$  is
- (b) If  $G$  is a group the center of  $G$  is
- (c) Let  $G$  be a group and  $S$  a nonempty subset of  $G$ . The subgroup generated by  $S$  is
- (d) Let  $G$  be a group and let  $x$  and  $y$  be elements of  $G$ . We say  $x$  and  $y$  are conjugate if

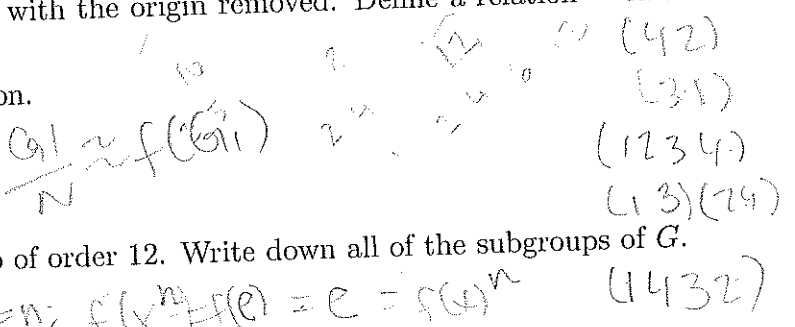
(20)2. Give examples of each of the following. No justification is required.

- (a) An infinite group in which every element has finite order. ✓
- (b) A nonidentity automorphism of  $S_3$ . ✓
- (c) A subgroup of  $S_4$  that is isomorphic to  $D_4$ . ✓
- (d) A noncyclic group  $G$  with exactly five subgroups (including  $G$  and  $\{e\}$ ). ✓

(10)3. Let  $S = \mathbb{R}^2 - (0,0)$ , the plane with the origin removed. Define a relation  $\sim$  on  $S$  by  $(a,b) \sim (c,d)$  if  $ad = bc$ .

- (a) Prove this is an equivalence relation.
- (b) Draw the equivalence classes.

$f \circ \pi = f$



(9)4. Let  $G = \langle g \rangle$  be a cyclic group of order 12. Write down all of the subgroups of  $G$ .

(9)5. True or False. No justification required.

- (a) Let  $G$  and  $K$  be finite groups and let  $f : G \rightarrow K$  be a group homomorphism. For all  $x \in G$ , the order of  $f(x)$  divides the order of  $x$ .
- (b) If  $S$  is a nonempty set with an associative binary operation for which there is an identity element, then the identity is unique.
- (c) In an abelian group every subgroup is normal.

$f(g, g^2) = f(g) f(g^2)$   
 $f(g, g^2) = (12)(6)(6)(12)$

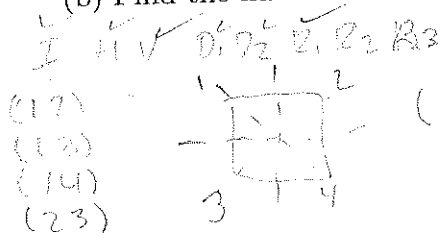
(10)6. Let  $G$  be a group. Call a subgroup  $H$  of  $G$  proper if  $H \neq G$ . Prove that if  $G$  is not the union of its proper subgroups, then  $G$  is cyclic.

$(1234)(567)$

$f(6) = (12)6(12)$

(10)7. (a) Find the largest possible order of an element of  $S_7$ .

(b) Find the number of elements in  $S_6$  that are conjugate to the following element:



$(123)(4567)$   
 $(12)(34)(56)(7)$   
 $(123)(45)$

$(12) \rightarrow (12)(12)(6) \rightarrow (6)$   
 $(13) \rightarrow (13)(13)(6) \rightarrow (6)$   
 $(23) \rightarrow (23)(12)(6) \rightarrow (6)$   
 $(123) \rightarrow (123)(12)(6) \rightarrow (6)$   
 $(132) \rightarrow (132)(12)(6) \rightarrow (6)$

$e \checkmark$   
 $(1246) \times \checkmark^{-1}$   
 $g_1 \in g_1 N (35) \checkmark^{-1}$   
 $g_2 \in g_1 N (1246)(35) \checkmark^{-1}$   
 $C_x = \{g \times g^{-1} \mid g \in S_6\}$

$(1 \ 2 \ 3 \ 4 \ 5 \ 6)$   
 $(2 \ 4 \ 5 \ 6 \ 3 \ 1)$

$6(1246)(35)6^{-1}$   
 $6(1246)(35)6^{-1} = 90$   
 $6! = 720$   
 $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$   
 $30 \cdot 3 = 90$

$|C_A| = \frac{|G|}{|C_G(x)|}$

(12)8. Let  $G_1$  and  $G_2$  be groups and let  $f: G_1 \rightarrow G_2$  be a homomorphism. Let  $N$  denote the kernel of  $f$ .

(a) Let  $\pi: G_1 \rightarrow G_1/N$  denote the canonical homomorphism. Describe  $\pi$  explicitly.

(b) Find a homomorphism  $\varphi: G_1/N \rightarrow G_2$  that satisfies the equation  $\varphi \circ \pi = f$ . Be sure to show your homomorphism is well defined.

(1) (a) Let  $G$  be a group and  $g \in G$ . The order of  $g$  is the smallest positive integer  $k$  s.t.  $g^k = e$ . If no such integer exists, then the order of  $g$  is infinite.

(b) If  $G$  is a group, the center of  $G$  is  $Z(G) = \{g \in G : gx = xg\}$ . All elements in  $G$  that commute with every other element.

(c) Let  $G$  be a group and  $S$  a nonempty subset of  $G$ . The subgroup generated by  $S$  is  $\langle S \rangle = \{s_1^{\pm 1} s_2^{\pm 1} \dots s_r^{\pm 1} \mid r > 0 \text{ and } s_i \in S\}$ .

$\langle S \rangle = \bigcap_{\substack{H \leq G \\ S \subseteq H}} H$ ; smallest subgroup of  $G$  that contains  $S$ .

(d) Let  $G$  be a group and let  $x$  or  $y$  be elements of  $G$ . We say  $x$  or  $y$  are conjugate if there exists  $z \in G$  such that  $z x z^{-1} = y$ .

(2) (a)  $(\mathbb{Q}, +) / (\mathbb{Z}, +)$ .

(b)  $f: S_3 \rightarrow S_3$ ; s.t.  $f$  is an isomorphism

(c) the subgroup  $H = \{e, (13)(24), (23), (14), (12)(34), (1243), (14)(23), (15421)\}$

$H \cong D_4$ .

(d) ?

$(1246)(1246)(35)(1246)$   
 $(21)(46) \{ (1246)(35) \} (21)(46)$   
 $(1246)(1246) = (62)(14)$

$gx = xg$   
 $x = g \times g^{-1}$