M403 Homework 1

Enrique Areyan
August 29, 2012

(1.1)

(i) **True**. Let C be an arbitrary nonempty set of negative integers. Define a new set D as follow:

$$D = \{d : d = -c \text{ for some } c \in C\}$$

By definition, D contains the additive inverses of C, hence $D \subseteq \mathbb{N}$ and $D \neq \emptyset$. Now, by the *Least Integer Axiom*, D has a smallest integer, call it n. Take $-n$ to obtain the largest integer in C.

(ii) **True**. $83, 84, 85, 86, ..., 95$. This sequence has 13 consecutive natural numbers and only 83 and 89 are prime.

(iii) **False**. $401, 402, 403, ..., 407$. This sequence has 7 consecutive natural numbers and only 401 is prime.

(iv) **True**. Let $C = \{l \in \mathbb{N} : l \text{ is the length of a sequence of consecutive natural numbers not containing 2 primes}\}$. Lengths here refer to the number of numbers in the sequence and hence, these are all natural numbers ($C \subseteq \mathbb{N}$). Also, $C \neq set$ (see (iii) above). Now, by the *Least Integer Axiom*, C has a smallest integer which correspond to the sequence of shortest length. Note that there may be more than one sequence of shortest length, but at least there is one.

(v) **True**. Using proposition 1.3, and the fact that $8 < \sqrt{79} < 9$, we need to check that 79 is not divisible by any prime between 2 and 8, i.e., 2,3,5,7. Check: $\frac{79}{2} = 39,5$; $\frac{79}{3} = 26,333...$; $\frac{79}{5} = 15,8$; $\frac{79}{7} = 11,2857143$.

(vi) **True**. Let $S(n) : n$ is an even number.

(vii) **False**. If $n = 2$, then $F_2 = F_1 + F_0 = 1 + 0 = 1$. Thus, $2 = n > F_2 = 1$

(viii) **False**. Let $m = 2$ and $n = 3$. Then $(m \cdot n)! = (2 \cdot 3)! = 6!$, but $2!3! = 2 \cdot 6 = 12 < 6!$

(1.2)

(i) For any $n \geq 0$ and any $r \neq 1$, prove that $1 + r + r^2 + ... + r^n = \frac{1-r^{n+1}}{1-r}$

Proof by induction: $S(n) : 1 + r + r^2 + ... + r^n = \frac{1-r^{n+1}}{1-r}$

Base Case: $S(0) : r^0 = 1 = \frac{1-r^{0+1}}{1-r} = \frac{1-r}{1-r} = 1 \Rightarrow S(0)$ is true.

Inductive Step: Assume that $S(n)$ is true. We want to show that $S(n+1)$ is true, i.e.,
$1 + r + r^2 + ... + r^n + r^{n+1} \stackrel{?}{=} \frac{1-r^{(n+1)+1}}{1-r}$. We begin as follow:

$$\begin{aligned} 1 + r + r^2 + ... + r^n + r^{n+1} &= \frac{1-r^{n+1}}{1-r} + r^{n+1} && \text{By inductive hypothesis} \\ = \frac{1-r^{n+1}+(r^{n+1})(1-r)}{1-r} &= \frac{1-r^{n+1}+r^{n+1}-r^{n+2}}{1-r} && \text{Summing fraction \& Collecting terms} \\ = \frac{1-r^{n+2}}{1-r} &= \frac{1-r^{(n+1)+1}}{1-r} && \text{Exponent rule} \end{aligned}$$

$\Longrightarrow S(n+1)$ is true.

(ii) Prove that $1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1$

Proof by induction: $S(n) : 1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1$

Base Case: $S(0) : 2^0 = 1 = 2^{0+1} - 1 = 2 - 1 = 1 \Rightarrow S(0)$ is true.

Inductive Step: Assume that $S(n)$ is true. We want to show that $S(n+1)$ is true, i.e.,
$1 + 2 + 2^2 + ... + 2^n + 2^{n+1} \stackrel{?}{=} 2^{(n+1)+1} - 1$. We begin as follow:

$$\begin{aligned} 1 + 2 + 2^2 + ... + 2^n + 2^{n+1} &= 2^{n+1} - 1 + 2^{n+1} && \text{By inductive hypothesis} \\ &= 2(2^{n+1}) - 1 && \text{Collecting terms} \\ &= 2^{(n+1)+1} - 1 && \text{Exponent rule} \end{aligned}$$

$\Longrightarrow S(n+1)$ is true.

(1.3) Show, for all $n \geq 1$, that 10^n leaves remainder of 1 after dividing by 9. (Note that we can formulate this as $10^n = 9p_n + 1$, where p_n is an integer)

Proof by induction: $S(n): 10^n = 9p_n + 1$

Base Case: $S(1): 10^1 = 9 \cdot 1 + 1 \Rightarrow S(1)$ is true.

Inductive Step: Assume that $S(n)$ is true. We want to show that $S(n+1)$ is true, i.e., $10^{n+1} \stackrel{?}{=} 9q_n + 1$, for some integer q_n. We begin as follow:

$$\begin{aligned}
10^{n+1} &= 10(10^n) && \text{Exponent rule}\\
&= 10(9p_n + 1) && \text{Inductive hypothesis}\\
&= 90p_n + 10 && \text{Distribution over integers}\\
&= 90p_n + 9 + 1 && \text{Since 10=9+1}\\
&= 9(10p_n + 1) + 1 && \text{Distribution over integers}\\
&= 9q_n + 1 && \text{where } q_n = (10p_n + 1) \text{ an integer}
\end{aligned}$$

$\Longrightarrow S(n+1)$ is true.

(1.5) Prove that $1^2 + 2^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$

Proof by induction: $S(n): 1^2 + 2^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$

Base Case: $S(1): 1 = 1^2 = \frac{1}{6}1 \cdot 2 \cdot 3 = \frac{6}{6} = \frac{1}{3}1^3 + \frac{1}{2}1^2 + \frac{1}{6}1 \Rightarrow S(1)$ is true.

Inductive Step: Assume that $S(n)$ is true. We want to show that $S(n+1)$ is true, i.e.,

$$1^2 + 2^2 + ... + n^2 + (n+1)^2 \stackrel{?}{=} \frac{1}{6}(n+1)(n+2)(2n+3)$$

We begin as follow:

$$\begin{aligned}
1^2 + 2^2 + ... + n^2 + (n+1)^2 &= \tfrac{1}{6}n(n+1)(2n+1) + (n+1)^2 && \text{Inductive Hypothesis}\\
&= \tfrac{n(n+1)(2n+1)+6(n+1)^2}{6} && \text{Summing fraction}\\
&= \tfrac{(n^2+n)(2n+1)+6n^2+12n+6}{6} && \text{Elementary arithmetic}\\
&= \tfrac{2n^3+n^2+2n^2+n+6n^2+12n+6}{6} && \text{Elementary arithmetic}\\
&= \tfrac{2n^3+9n^2+13n+6}{6} && \text{Elementary arithmetic}\\
&= \tfrac{1}{6}(n+1)(n+2)(2n+3) && \text{Distributive law}
\end{aligned}$$

$\Longrightarrow S(n+1)$ is true.

(1.6) Prove that $1^3 + 2^3 + ... + n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$

Proof by induction: $S(n): 1^3 + 2^3 + ... + n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$

Base Case: $S(1): 1 = 1^3 = \frac{1}{4}1^4 + \frac{1}{2}1^3 + \frac{1}{4}1^2 = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = \frac{1}{2} + \frac{1}{2} = 1 \Rightarrow S(1)$ is true.

Inductive Step: Assume that $S(n)$ is true. We want to show that $S(n+1)$ is true, i.e.,

$$1^3 + 2^3 + ... + n^3 + (n+1)^3 \stackrel{?}{=} \frac{1}{4}(n+1)^4 + \frac{1}{2}(n+1)^3 + \frac{1}{4}(n+1)^2$$

To make matters simpler, we can expand the right hand side of this equation to obtain a simpler expression:

$$\begin{aligned}
\tfrac{1}{4}(n+1)^4 + \tfrac{1}{2}(n+1)^3 + \tfrac{1}{4}(n+1)^2 &= \tfrac{1}{4}(n^2+2n+1)^2 + \tfrac{1}{2}((n+1)(n^2+2n+1)) + \tfrac{1}{4}(n^2+2n+1)\\
= \tfrac{1}{4}(n^4 + 2n^3 + n^2 + 2n^3 + 4n^2 + 2n + n^2 + 2n + 1) &+ \tfrac{1}{2}(n^3 + 2n^2 + n + n^2 + 2n + 1) + \tfrac{1}{4}(n^2 + 2n + 1)\\
&= \tfrac{1}{4}n^4 + n^3(1 + \tfrac{1}{2}) + n^2(\tfrac{6}{4} + \tfrac{3}{2} + \tfrac{1}{4}) + n(1 + \tfrac{3}{2} + \tfrac{1}{2}) + \tfrac{1}{4} + \tfrac{1}{2} + \tfrac{1}{4}\\
&= \tfrac{1}{4}n^4 + \tfrac{3}{2}n^3 + \tfrac{13}{4}n^2 + 3n + 1
\end{aligned}$$

Now we begin the inductive step as follow:

$$\begin{aligned}
1^3 + 2^3 + ... + n^3 + (n+1)^3 &= \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2 + (n+1)^3 && \text{Inductive Hypothesis}\\
&= \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2 + n^3 + 3n^2 + 3n + 1 && \text{Expanding } (n+1)^3\\
&= \tfrac{1}{4}n^4 + n^3(\tfrac{1}{2} + 1) + n^2(\tfrac{1}{4} + 3) + 3n + 1 && \text{Collecting terms}\\
&= \tfrac{1}{4}n^4 + \tfrac{3}{2}n^3 + \tfrac{13}{4}n^2 + 3n + 1 && \text{Elementary arithmetic}
\end{aligned}$$

$\Longrightarrow S(n+1)$ is true.

(1.8) The guess for the formula is $1+3+5+...+(2n-1)=n^2, n \geq 1$

Proof by induction: $S(n): 1+3+5+...+(2n-1)=n^2$

Base Case: $S(1): 1 = 1^2 \Rightarrow S(1)$ is true.

Inductive Step: Assume that $S(n)$ is true. We want to show that $S(n+1)$ is true, i.e.,

$$1+3+5+...+(2n-1)+(2n+1) \stackrel{?}{=} (n+1)^2$$

We begin as follow:

$$\begin{aligned} 1+3+5+...+(2n-1)+(2n+1) &= n^2+(2n+1) && \text{Inductive Hypothesis} \\ &= n^2+2n+1 && \text{Associativity} \\ &= (n+1)^2 && \text{Completing square} \end{aligned}$$

$\Longrightarrow S(n+1)$ is true.

(1.18) Prove that $F_n < 2^n$ for all $n \geq 0$, where $F_0, F_1, F_2, ...$ is the Fibonaccci sequence

Proof by induction: $F_n < 2^n$

Base Case: $S(0): F_0 = 0 < 1 = 2^0 \Rightarrow S(0)$ is true. Also, $S(1): F_1 = 1 < 2 = 2^1 \Rightarrow S(1)$ is true.

Inductive Step: Assume that $S(k)$ is true for $k < n$ (second form of induction). We want to show that $S(n)$ is true, i.e.,

$$F_n \stackrel{?}{<} 2^n$$

We begin as follow, if $n \geq 2$:

$$\begin{aligned} F_n = F_{n-1} + F_{n-2} &< 2^{n-1} + 2^{n-2} && \text{Inductive Hypothesis} \\ &= 2(2^{n-2}) + 2^{n-2} && \text{Exponent rule} \\ &= 3(2^{n-2}) \\ &< 4(2^{n-2}) \\ &= 2^2(2^{n-2}) \\ &= 2^n && \text{Exponent rule} \end{aligned}$$

$\Longrightarrow S(n)$ is true.