M403 Homework 12

Enrique Areyan
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1. I) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 8 & 7 & 2 & 1 & 4 & 3 \end{pmatrix} = (\ 1\ \ 5\ \ 2\ \ 6\ )(\ 3\ \ 8\ )(\ 4\ \ 7\ )$.

   Then $sgn(\alpha) = (-1)^{8-3} = (-1)^5 = -1$

   II) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 2 & 8 & 7 & 5 & 4 & 6 \end{pmatrix} = (\ 1\ \ 3\ \ 2\ )(\ 4\ \ 8\ \ 6\ \ 5\ \ 7\ )$.

   Then $sgn(\alpha) = (-1)^{8-2} = (-1)^6 = 1$

   III) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 5 & 6 & 7 & 8 & 1 \end{pmatrix} = (\ 1\ \ 3\ \ 2\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ )$.

   Then $sgn(\alpha) = (-1)^{8-1} = (-1)^7 = 1$

   IV) Let $\alpha = (\ 1\ \ 2\ )(\ 2\ \ 3\ )(\ 3\ \ 4\ )(\ 4\ \ 5\ )(\ 5\ \ 6\ ) \in S_{10}$. Then $sgn(\alpha) = (-1)^5 = -1$

   V) Let $\alpha = (\ 1\ \ 2\ \ 3\ \ 4\ \ 5\ )(\ 5\ \ 6\ \ 7\ \ 8\ ) \in S_{10} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 9 & 10 \end{pmatrix} =$
   $= (\ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ )(\ 9\ )(\ 10\ )$. Then $sgn(\alpha) = (-1)^{10-3} = (-1)^7 = -1$

   VI) Let $\alpha = (\ 1\ \ 5\ \ 9\ )(\ 2\ \ 6\ \ 10\ )(\ 4\ ) \in S_{10} = (\ 1\ \ 5\ \ 9\ )(\ 2\ \ 6\ \ 10\ )(\ 4\ )(\ 3\ )(\ 7\ )(\ 8\ )$.

   Then $sgn(\alpha) = (-1)^{10-6} = (-1)^4 = 1$

   VII) Let $\alpha = (\ 1\ \ 2\ )(\ 2\ \ 3\ )(\ 3\ \ 4\ )(\ 4\ \ 5\ )(\ 5\ \ 6\ ) \in S_8$. Then $sgn(\alpha) = (-1)^5 = -1$

   Is the same as IV) since the number of transpositions are the same.

   VIII) Let $\alpha = (\ 1\ \ 2\ \ 3\ \ 4\ \ 5\ )(\ 5\ \ 6\ \ 7\ \ 8\ ) \in S_8 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 \end{pmatrix} = (\ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ )$
   Then $sgn(\alpha) = (-1)^{8-1} = (-1)^7 = -1$

   IX) Let $\alpha = (\ 1\ \ 5\ \ 9\ )(\ 2\ \ 6\ \ 10\ )(\ 4\ ) \in S_{12} = (\ 1\ \ 5\ \ 9\ )(\ 2\ \ 6\ \ 10\ )(\ 4\ )(\ 3\ )(\ 7\ )(\ 8\ )(\ 11\ )(\ 12\ )$

   Then $sgn(\alpha) = (-1)^{12-8} = (-1)^4 = 1$

2. Given $\alpha_n \in S_n$, by proposition 2.35 we can write it as a product of k transpositions, i.e.:

$$\alpha = (\ i_1\ \ i_2\ )(\ i_3\ \ i_4\ )\cdots(\ i_{r-1}\ \ i_r\ )$$

   By proposition 2.27, the inverse of α is:

$$\alpha^{-1} = (\ i_r\ \ i_{r-1}\ )\cdots(\ i_4\ \ i_3\ )(\ i_2\ \ i_1\ )$$

   Clearly, both α and α^{-1} have the same number of transpositions, i.e., both have k transposition. By definition 2 of sign, $sgn(\alpha) = (-1)^k = sgn(\alpha^{-1})$

3. 2.23 Consider the complete factorizations of σ and σ'.

$$\sigma = \beta_1\beta_2\cdots\beta_t(j) \quad \text{and} \quad \sigma' = \beta_1\beta_2\cdots\beta_t$$

   Where $t \in \mathbb{N}$. Since $\sigma \in S_n, \sigma' \in S_X$ and $|S_n| = n, |S_X| = n-1$, by definition of sign we have that $sgn(\sigma) = (-1)^{n-(t+1)} = (-1)^{n-t-1} = (-1)^{(n-1)-t} = sgn(\sigma')$

2.26 Let $\alpha = \begin{pmatrix} i_1 & i_2 & \cdots & i_r \end{pmatrix} \in S_n$ be an r-cycle.

($\Rightarrow$) Assume that α is an even permutation. By definition, $sgn(\alpha) = 1 = (-1)^k$ where k is an even number, i.e, $k = 2l$ where $l \in \mathbb{N}$. We also know that α can be written as the product of k transpositions:

$$\alpha = \begin{pmatrix} i_1 & i_r \end{pmatrix} \begin{pmatrix} i_1 & i_{r-1} \end{pmatrix} \cdots \begin{pmatrix} i_1 & i_2 \end{pmatrix}$$

Since we fix i_1 for each transposition and vary i_j where $2 \leq j \leq r$, we can conclude that there are $r-1$ transpositions in the above decomposition. Hence, $r - 1 = k = 2l \Rightarrow r = 2l+1$, which means that r is an odd number.

($\Leftarrow$) Let r be an odd number, i.e., $r = 2l - 1$ where $l \in \mathbb{N}$. Again, we can write α as a product of transpositions:

$$\alpha = \begin{pmatrix} i_1 & i_r \end{pmatrix} \begin{pmatrix} i_1 & i_{r-1} \end{pmatrix} \cdots \begin{pmatrix} i_1 & i_2 \end{pmatrix}$$

There are $r-1 = 2l-2$ transpositions. If we compute the sign of α we get: $sgn(\alpha) = (-1)^{2l-2} = (-1)^{2l}(-1)^{-2} = 1$, which by definition means that α is an even permutation.

4. 2.36 i) **False**. $e(e(2,3),4) = e(8,4) = 8^4 = 4096 \neq e(2, e(3,4) = e(2,81) = 2^{81}$.

ii) **False** Consider the group $(S_3, \circ)$. Let $\alpha = \begin{pmatrix} 1 & 2 \end{pmatrix} \in S_3$ and $\beta = \begin{pmatrix} 2 & 3 \end{pmatrix} \in S_3$. Then $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \neq \begin{pmatrix} 1 & 3 & 2 \end{pmatrix} = \beta\alpha$. Hence, $(S_3, \circ)$ is a non-abelian group which shows that not all groups are abelian.

iii) **True**, $(\mathbb{R}^+, \cdot)$ is a group since $\cdot : \mathbb{R}^+ \times \mathbb{R}^+$ is a binary associative operation. Also $1 \in \mathbb{R}^+$ is the unity since $1 \cdot x = x \cdot 1 = x$ and each element $x \in \mathbb{R}^+$ has an inverse $x^{-1} = \frac{1}{x} \in \mathbb{R}^+$.

iv) **False**, since there is no identity element. The only element that works for identity is 0 since $0+x = x+0 = x$ but $0 \notin \mathbb{R}^+$

2.37 Given elements $a_1, a_2, ..., a_n$ not necessarily distinct in a group G, we wish to prove that the inverse of $a_1 \cdot a_2 \cdots a_n$ is $a_n^{-1} \cdots a_2^{-1} \cdot a_1^{-1}$. By definition of inverse, we need to prove that $(a_n^{-1} \cdots a_2^{-1} \cdot a_1^{-1}) \cdot (a_1 \cdot a_2 \cdots a_n) \stackrel{?}{=} e$, where e is the unit of the group G. We prove this as follow:

$$\begin{aligned}
(a_n^{-1} \cdots a_2^{-1} \cdot a_1^{-1}) \cdot (a_1 \cdot a_2 \cdots a_n) &= a_n^{-1} \cdots a_2^{-1} \cdot (a_1^{-1} \cdot a_1) \cdot a_2 \cdots a_n && \text{Associativity} \\
&= a_n^{-1} \cdots a_2^{-1} \cdot e \cdot a_2 \cdots a_n && \text{By definition of inverse} \\
&= a_n^{-1} \cdots a_2^{-1} \cdot (e \cdot a_2) \cdots a_n && \text{Associativity} \\
&= a_n^{-1} \cdots a_2^{-1} \cdot a_2 \cdots a_n && \text{By definition of unit} \\
&\vdots \quad \text{Applying the above steps } n-2 \text{ more times} \\
&= a_n^{-1} \cdot a_n \\
&= e && \text{By definition of inverse}
\end{aligned}$$

$\Rightarrow (a_n^{-1} \cdots a_2^{-1} \cdot a_1^{-1}) \cdot (a_1 \cdot a_2 \cdots a_n) = e$, which means that $(a_n^{-1} \cdots a_2^{-1} \cdot a_1^{-1})$ is the inverse of $(a_1 \cdot a_2 \cdots a_n)$. By proposition 2.45 it follows that this same inverse works on the right as well.

2.38 (i) Let $\alpha = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix} =$
$= \begin{pmatrix} 1 & 5 & 4 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} \Rightarrow sgn(\alpha) = (-1)^{5-2} = -1$, which means that α is an odd permutation. The order of α is the lcm of the lenght of its cycle in the complete factorization, i.e., $O(\alpha) = lcm(2,3) = 6$. The inverse is: $\alpha^{-1} = \begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 4 & 5 & 1 \end{pmatrix}$

(ii) For 2.22 $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 9 \end{pmatrix} \begin{pmatrix} 8 & 2 \end{pmatrix} \begin{pmatrix} 3 & 7 \end{pmatrix} \begin{pmatrix} 4 & 6 \end{pmatrix} \begin{pmatrix} 5 \end{pmatrix}$ and hence, the order is $lcm(2,2,2,2,1) = 2$.

For 2.28, we have that $f(0) \equiv 0 \pmod{11}$, $f(1) \equiv 1 \pmod{11}$, $f(2) \equiv 6 \pmod{11}$, $f(3) \equiv 0 \pmod{9}$, $f(4) \equiv 5 \pmod{11}$, $f(5) \equiv 0 \pmod{3}$, $f(6) \equiv 10 \pmod{11}$, $f(7) \equiv 0 \pmod{2}$, $f(8) \equiv 8 \pmod{11}$, $f(9) \equiv 4 \pmod{11}$, $f(10) \equiv 7 \pmod{11}$. We can write f as a permutation as follow:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 0 & 1 & 6 & 9 & 5 & 3 & 10 & 2 & 8 & 4 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 10 & 7 \end{pmatrix} \begin{pmatrix} 3 & 9 & 4 & 5 \end{pmatrix}$$

Hence, the order of f is $lcm(4,4) = 4$

2.39 (i) Since any permutation can be factored into disjoint r-cycles, it suffices to count them to get the total number of permutation of a particular order. In particular, to count the number of elements of order 2 we need only to count every possible 2-cycle since $\begin{pmatrix} i_1 & i_2 \end{pmatrix}^2 = (1)$ or product of disjoint 2 cycles since disjoint cycles have the property $\left[\begin{pmatrix} i_1 & i_r \end{pmatrix}\begin{pmatrix} i_1 & i_r \end{pmatrix}\right]^2 = \begin{pmatrix} i_1 & i_r \end{pmatrix}^2 \begin{pmatrix} i_1 & i_r \end{pmatrix}^2 = (1)(1) = (1)$, etc .

Hence, for S_5: there are $\frac{5 \cdot 4}{2} = 10$ one 2-cycles of, and $\frac{5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 2 \cot 2!} = \frac{120}{8} = 15$ two 2-cycles, so there are $10 + 15 = 25$ elements of order 2 in S_5.

For S_6, there are $\frac{6 \cdot 5}{2} = 15$ one 2-cycles, and $\frac{6 \cdot 5 \cdot 4 \cdot 3}{2 \cdot 2 \cdot 2!} = 45$ two 2-cycles, and $\frac{6!}{2 \cdot 2 \cdot 2 \cdot 3!} = 15$ three 2-cycles, so there are $15 + 15 + 45 = 75$ elements of order 2 in S_6.

(ii) Given S_n, the number of elements of order 2 in S_n is:

$$\sum_{i=1}^{k} \frac{n!}{2^i \cdot i! \cdot (n - 2i)!}$$

where $k \in \mathbb{N}$ is such that $n = 2k$ if n is even and $n = 2k + 1$ if n is odd. Note that if $n = 1$ then there are no 2-cycles.