M403 Homework 2

Enrique Areyan
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(1) Let b and h be the base and height of a rectangle respectively. Suppose the perimeter is $2b + 2h = 40$:

$$\begin{array}{lll} \frac{2b+2h}{2} & \geq & \sqrt{2b2h} & \text{Inequality of the means} \\ \frac{40}{2} & \geq & \sqrt{4bh} & \text{By hypothesis the perimeter is 40} \\ 20 & \geq & 2\sqrt{bh} & \text{Arithmetic} \\ 10 & \geq & \sqrt{bh} & \text{Multiplying by } \frac{1}{2} \text{ both sides} \\ 10^2 & \geq & bh & \text{since } ()^2 \text{ is a monotonic increasing function in } \mathcal{R}^+ \end{array}$$

Hence, the area of the rectangle can be at most $10^2 = 100$ exactly when $2b = 2h \iff b = h = 10$, i.e., an square.

(2) Let w, d and h be the width, depth and height of a rectangular box in 3-space. Suppose $w + d + h = 60$:

$$\begin{array}{llll} \frac{w+d+h}{3} & \geq & \sqrt[3]{wdh} & \text{Inequality of the means} \\ \frac{60}{3} & \geq & \sqrt[3]{wdh} & \text{By hypothesis} \\ 20 & \geq & \sqrt[3]{wdh} & \text{Arithmetic} \\ 20^3 & \geq & wdh & \text{since } ()^3 \text{ is a monotonic increasing function} \end{array}$$

Hence, the volume of the box can be at most $20^3 = 8,000$ exactly when $w = d = h = 20$, i.e., a cube.

(3) Let w, d and h be the width, depth and height of a rectangular box in 3-space. Suppose

$$2(wd + dh + wh) = 600 \Rightarrow wd + dh + wh = 300 :$$

$$\begin{array}{llll} \frac{wd+dh+wh}{3} & \geq & \sqrt[3]{wddhwh} & \text{Inequality of the means} \\ \frac{300}{3} & \geq & \sqrt[3]{wddhwh} & \text{By hypothesis} \\ 100 & \geq & \sqrt[3]{wddhwh} & \text{Arithmetic} \\ 100^3 & \geq & wddhwh & \text{since } ()^3 \text{ is a monotonic increasing function} \\ 1,000,000 & \geq & (wdh)^2 & \text{Arithmetic and exponent rule} \\ \sqrt{1,000,000} & \geq & wdh & \text{since } \sqrt{()} \text{ is a monotonic increasing function in } \mathcal{R}^+ \end{array}$$

Hence, the volume of the box can be at most $\sqrt{1,000,000} = 1,000$ exactly when $wd = dh = wh \iff w = d = h = 10$, i.e., a cube.

(1.17) Prove that every positive integer n has a unique factorization $n = 3^k m$, where $k \geq 0$ and m is not multiple of 3.

Proof by 2nd form of induction: $S(n) : n = 3^k m$, where $k \geq 0$ and m is not multiple of 3.

Base Case: $S(1) : 1 = 3^0 \cdot 1$, 1 is not multiple of 3. This is true.

Inductive Step: Assument that $S(k)$ is true for every $k < n$. We want to show that $S(n)$ is true.

(i) If n is not multiple of 3, then: $n = 3^0 n, k = 0, m = n$; and we are done.

(ii) Otherwise, if n is multiple of 3, then:

$$\begin{array}{lll} n & = & 3a & \text{for some } a \text{ such that } 1 \leq a < n \\ & = & 3(3^p q) & \text{By inductive hypothesis, where } p \geq 0 \text{ and } q \text{ not multiple of 3} \\ & = & 3^{p+1} q & \text{Exponent rule, } k = p + 1 \text{ and } m = q \text{ not multiple of 3} \end{array}$$

To show that this decomposition is unique, suppose that it is not, i.e., that there exists $k' \geq 0$ and m' not multiple of 3, such that $n = 3^{k'} m'$ and $(k, m) \neq (k', m')$. Either $k' = 0$, in which case n is not multiple of 3 which is impossible or $k' \geq 1 \Rightarrow 3a = n = 3^{k'} m' \Rightarrow a = 3^{k'-1} m'$, where a was $a = 3^p q = 3^{k-1} m$. Hence, a has two different decompositions which is a contradiction to the proposition holding for $a < n$. Q.E.D

(1.19) If F_n denotes the nth term of the Fibonacci sequence, prove that

$$\sum_{n=1}^{m} F_n = F_{m+2} - 1$$

Proof: is by induction on m. $S(m) : \sum_{n=1}^{m} F_n = F_{m+2} - 1$

Base Case: $S(1) : F_1 = 1 = 2 - 1 = F_3 - 1$. Base case holds true.

Inductive Step: Assume $S(m)$ is true. We want to show that $S(m+1)$ is true, i.e., $\sum_{n=1}^{m+1} F_n \stackrel{?}{=} F_{(m+1)+2} - 1$

$$\begin{aligned}
\textstyle\sum_{n=1}^{m+1} F_n &= (\textstyle\sum_{n=1}^{m} F_n) + F_{m+1} && \text{Separating the sum} \\
&= (F_{m+2} - 1) + F_{m+1} && \text{By inductive hypothesis} \\
&= F_{m+2} + F_{m+1} - 1 && \text{Commutativity \& associativity} \\
&= F_{m+3} - 1 && \text{Definition of Fibonacci sequence} \\
&= F_{(m+1)+2} - 1 && \text{Q.E.D}
\end{aligned}$$

(1.30) Show that $\binom{n}{r} = \binom{n}{n-r}$.

$$\textbf{Proof}: \binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-n+r)!(n-r)!} = \frac{n!}{(n-(n-r))!(n-r)!} = \binom{n}{n-r}$$

(1.31) Show that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} = 2^n$

By corollary 1.19, for any real number x and for all integers $n \geq 0$,

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

Simply set $x = 1$, i.e.:

$$2^n = (1+1)^n = \sum_{r=0}^{n} \binom{n}{r} 1^r = \binom{n}{0} 1^0 + \binom{n}{1} 1^1 + \binom{n}{2} 1^2 + ... + \binom{n}{n} 1^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n}$$

(1.32)

(i) Using $x = -1$ in corollary 1.19 we obtain:

$$0 = 0^n = (1-1)^n = \sum_{r=0}^{n} \binom{n}{r} (-1)^r = \binom{n}{0} (-1)^0 + \binom{n}{1} (-1)^1 + \binom{n}{2} (-1)^2 + ... + \binom{n}{n} (-1)^n =$$

$$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - ... + (-1)^n \binom{n}{n}$$

(ii) Just move all odd (negative) terms to the right-hand side of the equation:

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - ... + (-1)^n \binom{n}{n} = 0 \iff \binom{n}{0} + \binom{n}{2} + ... + \binom{n}{n-1} = \binom{n}{1} + \binom{n}{3} + ... + \binom{n}{n}$$

$$\iff \binom{n}{r} = \binom{n}{r'}, \quad \text{where } r \text{ is even and } r' \text{ is odd}$$