M403 Homework 3

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(1.28)

(i) **True**. $\binom{7}{1} = \binom{7}{6} = 7; \binom{7}{2} = \binom{7}{5} = 7 \cdot 3; \binom{7}{3} = \binom{7}{4} = 7 \cdot 5$

(ii) **False**. For $n = 10$ and $r = 2$ we have that: $\binom{10}{2} = \frac{10!}{8!2!} = \frac{90}{2} = 45$ is not a multiple of $n = 10$

(iii) **True**. There are $\binom{10}{4}$ quartets of dogs and $\binom{10}{6}$ sextets of cats. By symmetry $\binom{10}{4} = \binom{10}{6}$

(vi) **True**. A direct consequence of corollary 1.28 setting $q = \frac{k}{n}$

(v) **True**. Let $f(x) = ax^2 + bx + c$. Let z be a complex number such that $f(z) = az^2 + bz + c = 0$. Then,

$$0 = \overline{0} = \overline{az^2 + bz + c} = \overline{az^2} + \overline{bz} + \overline{c} = \overline{a}\overline{z^2} + \overline{b}\overline{z} + \overline{c}$$

But, if a, b, c are real numbers, then $a = \bar{a}; b = \bar{b}; c = \bar{c}$, hence, $0 = a\overline{z^2} + b\overline{z} + c = f(\overline{z}) \iff \bar{z}$ is a root of $f(x)$

(vi) **False**. Let $f(x) = 0x^2 + ix + 1$. Then i is a root of $f(x)$ since $f(i) = i^2 + 1 = -1 + 1 = 0$. But, $f(\overline{i}) = f(-i) = -i^2 + 1 = 1 + 1 = 2$. Hence $\overline{i}$ is not a root of $f(x)$.

(vii) **True**. Because $i^4 = i^2 i^2 = -1 \cdot -1 = 1$ and $i^1 = i, i^2 = -1, i^3 = -i$. $(-i)^4 = (-i)^2(-i)^2 = -1 \cdot -1 = 1$ and $(-i)^1 = -i, (-i)^2 = -1, (-i)^3 = i$.

(1.34)

$$\begin{array}{rcll} \frac{n}{r}\binom{n-1}{r-1} & = & \frac{n}{r}\left(\frac{(n-1)!}{(n-1-(r-1))!(r-1)!}\right) & \text{Pascal definition} \\ & = & \frac{n(n-1)!}{(n-1-r+1)!r(r-1)!} & \text{Multiplication Associativity and Commutativity} \\ & = & \frac{n!}{(n-r)!r!} & \text{Arithmetics and definition of Factorial} \\ & = & \binom{n}{r} & \text{Pascal Definition} \end{array}$$

(1.39) Let X be a set with n elements.

(i) To count the number of subsets of X we need to count all possible subsets of size r where $0 \leq r \leq n$, i.e., subsets of size 0 (empty set), subset of size one (singleton), ..., and finally the set X itself. This is exactly the different ways of choosing zero elements from the set, then choosing a single element and so on, and then summing these numbers. It was proven in the last homework that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} = 2^n$, which is the expression we need. Hence, the number of subsets of X is 2^n. Q.E.D.

(ii) Counting the number of subsets of X is equivalent to counting the number of different bit strings of length n. To see why this is the case, first arrange the elements of the set X, in a linear manner. To construct a subset, take a bit string of length n. A 1 in position k in the bit string indicates that the kth element of X should be included in the subset. Likewise, a 0 indicates that the element should be excluded. All possible subset of X can be constructed in this way. Hence, to count the possible number of subset it suffices to count the number of bit strings.
To construct a bit string of length n we can choose two possibilites for each position, i.e., either a 1 or 0. Hence, there are $2 \cdot 2 \cdot 2 \cdot ... \cdot 2 = 2^n$ bit strings or subsets of X.

(1.40) There are $\binom{45}{5}$ different lottery tickets. To be the winner is to have one ticket, hence $\frac{1}{\binom{45}{5}} = 8.18492027 \cdot 10^{-7}$ is the probability of winning.

(1.43) First, note that the number i in polar coordinates is just: $(1, \frac{\pi}{2})$. We want to find a complex number z such that $z^2 = i$, i.e., $(r, \theta)^2 = (1, \frac{\pi}{2})$. By DeMoivre's theorem, $(r, \theta)^2 = (r^2, 2 \cdot \theta)$, Hence:

$$(r^2, 2 \cdot \theta) = \begin{cases} r^2 = 1 & \Rightarrow r = 1 \text{ we take just the positive one} \\ 2\theta = \frac{\pi}{2} + 2\pi k & \text{for } k = 0, 1 \Rightarrow \theta = \frac{\pi}{4} \text{ OR } \theta = \frac{5\pi}{4} \end{cases}$$

The two 2-roots of i in polar coordinates are $(1, \frac{\pi}{4})$ and $(1, \frac{5\pi}{4})$. We can easily check that, by De Moivre's Theorem, $(1, \frac{\pi}{4})^2 = (1^2, 2\frac{\pi}{4}) = (1, \frac{\pi}{2})$ and $(1, \frac{5\pi}{4})^2 = (1^2, 2\frac{5\pi}{4}) = (1, \frac{\pi}{2})$

(1.44)

(i) We want to show that $w^n \stackrel{?}{=} z$.

$$\begin{aligned} w^n &= (\sqrt[n]{r}[cos(\theta/n) + isin(\theta/n)])^n && \text{By definition of } w \\ &= (\sqrt[n]{r})^n[cos(\tfrac{\theta}{n}n) + isin(\tfrac{\theta}{n}n)] && \text{De Moivre's Theorem and law of exponent} \\ &= r[cos(\theta) + isin(\theta)] && \text{Arithmetic} \\ &= z && \text{By definition of } z \end{aligned}$$

(ii) Let ζ be a primitive nth root of unity. We want to show that $(\zeta^k w)^n \stackrel{?}{=} z$.

$$\begin{aligned} (\zeta^k w)^n &= (\zeta^k)^n w^n && \text{Exponent law} \\ &= (\zeta^n)^k z && \text{Exponent law and } w^n = z \text{ by part (i)} \\ &= 1^k z && \text{By hypothesis} \\ &= z && \text{Q.E.D} \end{aligned}$$

(1.45) To find the nth root of a complex number we use De Moivre's Theorem and set if $z = r(cos\theta + isin\theta)$ then

$$\sqrt[n]{z} = \sqrt[n]{r}\left[cos\left(\frac{\theta + 2\pi k}{n}\right) + isin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

For $k = 0, 1, ..., n-1$. Equivalently, we could have used 360° instead of 2π

(i) If $z = 8 + 15i$, then $r = \sqrt{8^2 + 15^2} = \sqrt{64 + 225} = \sqrt{289} = 17$. By potting this number in the complex plane and use basic trigonometry, we find that $\theta = arctan(\frac{15}{8}) = 61.927°$. Hence, the two roots are:

$$\begin{aligned} \sqrt{17}\left[cos\left(\tfrac{61.927° + 360° \cdot 0}{2}\right) + isin\left(\tfrac{61.927° + 360° \cdot 0}{2}\right)\right] &= \sqrt{17}\,[cos(30.9635°) + isin(30.9635°)] \\ &= (\sqrt{17}, 30.9635°) \end{aligned}$$

$$\begin{aligned} \sqrt{17}\left[cos\left(\tfrac{61.927° + 360° \cdot 1}{2}\right) + isin\left(\tfrac{61.927° + 360° \cdot 1}{2}\right)\right] &= \sqrt{17}\,[cos(210.9635°) + isin(210.9635°)] \\ &= (\sqrt{17}, 210.9635°) \end{aligned}$$

(ii) Applying the same reasoning as before:

$$\begin{aligned} \sqrt[4]{17}\left[cos\left(\tfrac{61.927° + 360° \cdot 0}{4}\right) + isin\left(\tfrac{61.927° + 360° \cdot 0}{4}\right)\right] &= \sqrt[4]{17}\,[cos(15.48175°) + isin(15.48175°)] \\ &= (\sqrt[4]{17}, 15.48175°) \end{aligned}$$

$$\begin{aligned} \sqrt[4]{17}\left[cos\left(\tfrac{61.927° + 360° \cdot 1}{4}\right) + isin\left(\tfrac{61.927° + 360° \cdot 1}{4}\right)\right] &= \sqrt[4]{17}\,[cos(105.48175°) + isin(105.48175°)] \\ &= (\sqrt[4]{17}, 105.48175°) \end{aligned}$$

$$\begin{aligned} \sqrt[4]{17}\left[cos\left(\tfrac{61.927° + 360° \cdot 2}{4}\right) + isin\left(\tfrac{61.927° + 360° \cdot 2}{4}\right)\right] &= \sqrt[4]{17}\,[cos(195.48175°) + isin(195.48175°)] \\ &= (\sqrt[4]{17}, 195.48175°) \end{aligned}$$

$$\begin{aligned} \sqrt[4]{17}\left[cos\left(\tfrac{61.927° + 360° \cdot 3}{4}\right) + isin\left(\tfrac{61.927° + 360° \cdot 3}{4}\right)\right] &= \sqrt[4]{17}\,[cos(285.48175°) + isin(285.48175°)] \\ &= (\sqrt[4]{17}, 285.48175°) \end{aligned}$$

1. Show the triangle inequality: for any complex numbers z and w, $|z + w| \stackrel{?}{\leq} |z| + |w|$.
Proof: Let $z, w \in \mathbb{C}$.

$$\begin{aligned} |z+w|^2 &= (z+w)\overline{(z+w)} && \text{By definition of multiplication of complex conjugates} \\ &= (z+w)(\bar{z} + \bar{w}) && \text{Properties of conjugation} \\ &= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} && \text{Distributivity} \\ &= |z|^2 + |w|^2 + z\bar{w} + \overline{z\bar{w}} && \text{Complex conjugation and multiplication} \\ &= |z|^2 + |w|^2 + 2 \cdot RealPart(z\bar{w}) && \text{Since summing a number by its conjugate eliminates de complex part} \\ &\leq |z|^2 + |w|^2 + 2|z\bar{w}| && \text{Absolute values are at least as big} \\ &= |z|^2 + |w|^2 + 2|z||\bar{w}| && \text{Property of absolute values} \\ &= (|z| + |w|)^2 && \text{Arithmetic} \end{aligned}$$

Hence

$$|z+w|^2 \leq (|z| + |w|)^2 \iff \text{(taking square root) } |z+w| \leq |z| + |w|$$