Name:

# M403 - Fall 2011 - Dr. A. Lindenstrauss.

## MIDTERM 1

20 pts

(1) Let $F_i$ be the i'th Fibonacci number, $F_0=0$, $F_1=1$, and $F_n=F_{n-1}+F_{n-2}$ for $n\geq 2$.
Prove that $F_n \geq n-1$ for all $n\geq 0$
(Set up the proof in detail, make sure every case is covered, explain every step)

Check directly:

$$F_0 = 0 \overset{?}{\geq} 0-1=-1 \checkmark$$
$$F_1 = 1 \overset{?}{\geq} 1-1=0 \checkmark$$
$$F_2 = 1 \overset{?}{\geq} 2-1=1 \checkmark$$
$$F_3 = 2 \overset{?}{\geq} 3-1=2 \checkmark$$

~~Induc~~ Claim: $F_n \geq n-1$ for all $n\geq 0$

Base cases: checked above

Inductive step: Assume that $F_k \geq k-1$ for all $0\leq k<n$ & $n\geq 4$. Show that $F_n\geq n-1$

By definition, $F_n = F_{n-1}+F_{n-2}$.

By the inductive hypothesis, $F_{n-1}\geq n-2$ & $F_{n-2}\geq n-3$.

So $F_n = F_{n-1}+F_{n-2} \geq (n-2)+(n-3) = 2n-5 =$

$= (n-1)+(n-4) \geq n-1$

because $n\geq 4 \Rightarrow n-4\geq 0$.

Note: This particular inductive step works for $n\geq 4$, so I had to check the $n<4$ cases by hand.

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(2) Prove that $2^2+4^2+\dots+(2n)^2 = \frac{2n(n+1)(2n+1)}{3}$ for every $n \geq 1$ (set up the proof in detail, explain every step).

Proof by induction:

Base case: $n=1$ $\quad 2^2 \stackrel{?}{=} \frac{2\cdot(1+1)(2+1)}{3} = 4$ ✓

Inductive step: Assume $n \geq 2$ and assume by induction $2^2+4^2+\dots+(2(n-1))^2 = \frac{2(n-1)n(2n-1)}{3}$ and show that $2^2+4^2+\dots+(2n)^2 \stackrel{?}{=} \frac{2n(n+1)(2n+1)}{3}$

$$2^2+4^2+\dots+(2n)^2 = (2^2+4^2+\dots+(2(n-1))^2 + (2n)^2 =$$

$$\underset{\substack{\uparrow \\ \text{by inductive hypothesis}}}{=} \frac{2(n-1)n(2n-1)}{3} + (2n)^2 = 2n\left(\frac{(n-1)(2n-1)}{3} + 2n\right) =$$

$$= 2n\,\frac{2n^2-3n+1+6n}{3} = \frac{2n(2n^2+3n+1)}{3} = \frac{2n(n+1)(2n+1)}{3}$$ ✓

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(3) Prove that for any two positive numbers $a$ & $b$ $\frac{a+b}{2} \geq \sqrt{ab}$. (You may not use the inequality of the means! That is what you are supposed to show. Justify carefully).

When do you get equality? Justify.

Since both sides are positive, the inequality (and corresponding equality) hold iff they do for

$$\left(\frac{a+b}{2}\right)^2 \stackrel{?}{\geq} (\sqrt{ab})^2 = ab$$

$$\iff \frac{a^2+2ab+b^2}{4} \geq ab$$

$$\iff a^2+2ab+b^2 \geq 4ab$$

$$\iff a^2-2ab+b^2 \geq 0$$

$$\iff (a-b)^2 \geq 0$$ and this always holds since $x^2 \geq 0$ for all $x \in \mathbb{R}$.

Equality happens $\iff a-b=0$, that is: IFF $a=b$

2.

(4) Say you have a list of claims $S(n,r)$ for $n \geq 0$, $0 \leq r \leq n$. Say you also know that:

(i) $S(n,0)$ is true for every $n \geq 0$.

(ii) $S(n,n)$ is true for every $n \geq 0$.

(iii) If for some $n$ & $r$, ~~[illegible]~~ $n \geq 0$, $1 \leq r \leq n-1$, both $S(n-1, r-1)$ AND $S(n-1, r)$ are true, THEN $S(n,r)$ is true.

Prove that $S(n,r)$ is true for all $n \geq 0$, $0 \leq r \leq n$.

Let $T(n)$ be the claim: $S(n,r)$ is true for all $0 \leq r \leq n$.

Prove $T(n)$ for all $n \geq 0$ by induction on $n$.

Base case: $n=0$: $T(0)$ just asserts that $S(0,0)$ and that follows both from (i) and from (ii).

Inductive step: Assume that $n \geq 1$ and $T(n-1)$ is true. Show that $T(n)$ is true:

I need to check $S(n,r)$ for $0 \leq r \leq n$:

a) $S(n,0)$ is true by (i). This is the $r=0$ case.

b) If $1 \leq r \leq n-1$, Since $T(n-1)$ is true I know that $S(n-1, s)$ is true for all $0 \leq s \leq n-1$. In particular, both $r$ & $r-1$ lie between $0$ and $n-1$ so $S(n-1, r-1)$ and $S(n-1, r)$ are both true. By (iii), this implies that $S(n,r)$ is true.

c) $S(n,n)$ is true by (ii). This is the $r=n$ case.

So $S(n,r)$ is true for all $0 \leq r \leq n$.

5 pts each

(5) Short answer - no justification required
For the questions asking about complex numbers, write them as $re^{i\theta}$, $0 \le r$, $0 \le \theta < 2\pi$

a) List all the 8'th roots of unity.

$1,\ e^{\frac{2\pi i}{8}},\ e^{\frac{2\pi i}{8}\cdot 2},\ e^{\frac{2\pi i}{8}\cdot 3},\ e^{\frac{2\pi i\cdot 4}{8}},\ e^{\frac{2\pi i\cdot 5}{8}},\ e^{\frac{2\pi i\cdot 6}{8}},\ e^{\frac{2\pi i\cdot 7}{8}}$

$e^{\frac{\pi}{4}i} \quad e^{\frac{\pi}{2}i} = i \quad e^{\frac{3\pi}{4}i} \quad e^{\pi i} = -1 \quad e^{\frac{5\pi}{4}i} \quad e^{\frac{3\pi}{2}i} = -i \quad e^{\frac{7\pi}{4}i}$

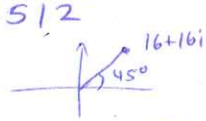
b) List the primitive 8'th roots of unity.

$e^{\pi/4\, i},\ e^{3\pi/4\, i},\ e^{5\pi/4\, i},\ e^{7\pi/4\, i}$

c) What is $\sqrt{16+16i}$ ?

$|16+16i| = \sqrt{256+256} = \sqrt{512}$

The direction angle of $16+16i$ is $\frac{\pi}{4}$




so $16+16i = \sqrt{512}\cdot e^{i\pi/4}$

$\sqrt{16+16i} = \pm\sqrt[4]{512}\, e^{i\pi/8} = \pm 4\sqrt[4]{2}\, e^{i\pi/8}$

d) What is the coefficient of $x^2$ in $(1+x)^{100}$ ?

$$\binom{100}{2} = \frac{100\cdot 99}{1\cdot 2} = 4950$$

according to the specified format, these should be written as

$4\sqrt[4]{2}\, e^{i\pi/8}$

$4\sqrt[4]{2}\, e^{i9\pi/8}$