SOLUTIONS Name:

M403 - Fall 2012 - Dr. A. Lindenstrauss

MIDTERM I

20 pts

① Let F_i be the i'th Fibonacci number, $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.
Prove that $F_n > (1.2)^n$ for all $n \geq 3$.
You can use: $(1.2)^2 = 1.44$, $(1.2)^3 = 1.728$, $(1.2)^4 = 2.0736$.
You should not need to do any complicated calculations.
Set up your induction carefully - explain all the details.

Proof by induction on n (2nd kind)

Base cases: $n = 3$ $\quad (1.2)^3 = 1.728 < 2 = F_3$ ✓

$n = 4$ $\quad (1.2)^4 = 2.0736 < 3 = F_4$ ✓

Inductive step: Assume $n \geq 5$ and $F_k > (1.2)^k$ for all $3 \leq k < n$ and show that $F_n > (1.2)^n$:

$$F_n \overset{\text{def}}{=} F_{n-1} + F_{n-2} \underset{\substack{\text{inductive hypothesis} \\ \text{for } n-1 \text{ AND for } n-2}}{>} (1.2)^{n-1} + (1.2)^{n-2} =$$

$$= (1.2)^{n-2}(1.2 + 1) = (1.2)^{n-2} \cdot (2.2)$$

$$> (1.2)^{n-2} \cdot (1.44) = (1.2)^n.$$

$F_n > (1.2)^n$ ✓

10 points

(2) a) Prove that there is only one way to divide with remainder by an integer $a>0$, that is: if $b \in \mathbb{Z}$ and you can write $b = qa + r = q'a + r'$ with $q, q', r, r' \in \mathbb{Z}$, $0 \leq r < a$, $0 \leq r' < a$: then it must be true that $q = q'$ and $r = r'$.

If $b = qa + r = q'a + r'$ then

$$(q - q')a = r' - r$$

Now both r & r' are between 0 and $a-1$

so $|r' - r| < a$. So $|q - q'| \cdot a < a$ but $|q - q'| \in \mathbb{N}$

so either it is zero or it is ≥ 1, which is impossible because $|q - q'| \cdot a < a \Rightarrow |q - q'| < 1$. So $|q - q'| = 0$,

$\Rightarrow q - q' = 0$, $\Rightarrow r' - r = (q - q')a = 0$. $\Rightarrow \begin{cases} q = q' \\ r = r' \end{cases}$

20 points

b) Use your answer in a) to show that there are infinitely many different primes.

Proof by contradiction: Say there was a finite list of primes, $p_1, p_2, \ldots, p_n$.

Let $p = p_1 p_2 \cdots p_n + 1$. For each p_i, $p = q_i \cdot p_i + 1$ for q_i the product of all the p_k except p_i.

Therefore by a), we cannot write $p = \tilde{q}_i \, p_i + 0$ for $\tilde{q}_i \in \mathbb{Z}$, so $p_i \nmid p$. But then p is neither a prime (then it would be one of the p_i & divide itself) nor a product of primes, although it is an integer ≥ 2, in contradiction to the theorem that said this was impossible.

15 points

(3) Prove that for every $n \geq 1$, $3+6+9+\cdots+3n = \frac{3n(n+1)}{2}$.
Set up the proof in detail, and explain every step.

Base Case: $n=1$: $3 \stackrel{?}{=} \frac{3 \cdot 1 \cdot 2}{2}$

Inductive Step: Assume $3+6+9+\cdots+3n = \frac{3n(n+1)}{2}$

show $3+6+9+\cdots+3n+3(n+1) \stackrel{?}{=} \frac{3(n+1)(n+2)}{2}$

$$3+6+9+\cdots+3n+3(n+1) = [3+6+9+\cdots+3n]+3(n+1)$$

$$\underset{\substack{\uparrow \\ \text{inductive} \\ \text{hypothesis}}}{=} \frac{3n(n+1)}{2} + 3(n+1) = \frac{(3n+6)(n+1)}{2} = \frac{3(n+1)(n+2)}{2}$$

15 points

(4) Prove that any integer $n \geq 2$ is either a prime or a product of primes. Explain each step carefully.

Proof by induction on n (2nd kind)

Base case: 2 is prime.

Inductive step: Assume $n \geq 2$ and for every $2 \leq k < n$, either k is prime or it is a product of primes.

Consider n itself. Either it is prime, and we are done, or it has a factor $a \notin \{\pm 1, \pm n\}$ so there is $b \in \mathbb{Z}$ with $ab = n$. Since $n > 0$, WLOG $a, b > 0$ (otherwise look at $-a$ & $-b$. $a, b \neq 0$ because their product is n). But $\begin{cases} a \neq 1 \\ a \neq n \end{cases}$ so $n > a > 1$ so $2 \leq b < n$, and $\begin{cases} a \neq 1 \\ a \neq n \end{cases}$ so $2 \leq a < n$. By the inductive hypothesis a & b are primes or products of primes. So n is the product of the decompositions of a & b into primes.

5 points each

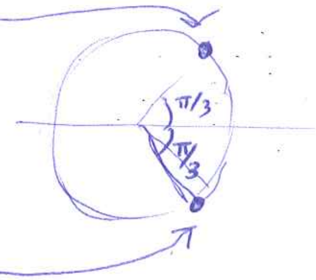
Short answer - no justification required.
For the questions asking about complex numbers, write the answers as $re^{i\theta}$, $0 \leq r$, $0 \leq \theta < 2\pi$.

a) What are all the $z \in \mathbb{C}$ with $z^6 = 64$?

$2, \ 2e^{\frac{\pi}{3}i}, \ 2e^{\frac{2\pi}{3}i}, \ 2e^{\pi i} = -2, \ 2e^{\frac{4\pi}{3}i}, \ 2e^{\frac{5\pi}{3}i}$

b) What are all the primitive 6'th roots of unity?

$e^{\frac{2\pi}{6}i} = e^{\frac{\pi}{3}i}$ $\qquad$ $e^{\frac{10\pi}{6}i} = e^{\frac{5}{3}\pi i}$



c) What is the coefficient of x^{19} in $(1+x)^{21}$?

$\binom{21}{19} = \frac{21!}{19!\,2!} = \frac{21 \cdot 20}{1 \cdot 2} = 210$

d) $\zeta = e^{\frac{8\pi i}{12}}$ is a primitive k'th root of unity for some k. What is k?

$\zeta = e^{2\pi i \cdot \frac{4}{12}} = e^{2\pi i \cdot \frac{1}{3}}$ $\qquad$ $\underline{k = 3}$