TEST FOR POSITIVE AND NEGATIVE DEFINITENESS

We want a computationally simple test for a symmetric matrix to induce a positive definite quadratic form. We first treat the case of 2×2 matrices where the result is simple. Then, we present the conditions for $n \times n$ symmetric matrices to be positive definite. Finally, we state the corresponding condition for the symmetric matrix to be negative definite or neither. Before starting all these cases, we recall the relationship between the eigenvalues and the determinant and trace of a matrix.

For a matrix $\mathbf{A}$, the determinant and trace are the product and sum of the eigenvalues:

$$\det(\mathbf{A}) = \lambda_1 \cdots \lambda_n, \qquad \text{and}$$
$$\operatorname{tr}(\mathbf{A}) = \lambda_1 + \cdots + \lambda_n,$$

where λ_j are the n eigenvalues of $\mathbf{A}$. (Here we list an eigenvalue twice if it has multiplicity two, etc.)

1. Two by two matrices

Let $\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ be a general 2×2 symmetric matrix. We will see in general that the quadratic form for $\mathbf{A}$ is positive definite if and only if all the eigenvalues are positive. Since, $\det(\mathbf{A}) = \lambda_1\lambda_2$, it is necessary that the determinant of $\mathbf{A}$ be positive. On the other hand if the determinant is positive, then either (i) both eigenvalues are positive, or (ii) both eigenvalues are negative. Since $\operatorname{tr}(\mathbf{A}) = \lambda_1 + \lambda_2$, if $\det(\mathbf{A}) > 0$ and $\operatorname{tr}(\mathbf{A}) > 0$ then both eigenvalues must be positive. We want to give this in a slightly different form that is more like what we get in the $n \times n$ case. If $\det(\mathbf{A}) = ac - b^2 > 0$, then $ac > b^2 \geq 0$, and a and c must have the same sign. Thus $\det(\mathbf{A}) > 0$ and $tr(\mathbf{A}) > 0$ is equivalent to the condition that $\det(\mathbf{A}) > 0$ and $a > 0$. Therefore, a necessary and sufficient condition for the quadratic form of a symmetric 2×2 matrix to be positive definite is for $\det(\mathbf{A}) > 0$ and $a > 0$.

We want to see the connection between the condition on $\mathbf{A}$ to be positive definite and completion of the squares.

$$\begin{aligned} \mathbf{Q}(x, y) &= (x, y)\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= a\,x^2 + 2b\,xy + c\,y^2 \\ &= a\Big(x + \frac{b}{a}y\Big)^2 + \Big(\frac{ac - b^2}{a}\Big)y^2. \end{aligned}$$

This expresses the quadratic form as a sum of two squares by means of "completion of the squares". If $a > 0$ and $\det(\mathbf{A}) > 0$, then both these coefficients are positive and the form is positive definite. It can also be checked that a and $\dfrac{ac - b^2}{a}$ are the pivots when $\mathbf{A}$ is row reduced. We can summarize these two results in the following theorem.

Theorem 1. *Let $\mathbf{A}$ be an 2×2 symmetric matrix and $\mathbf{Q}(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ the related quadratic form. The following conditions are equivalent:*

(i) $\mathbf{Q}(\mathbf{x})$ *is positive definite.*
(ii) *Both eigenvalues of $\mathbf{A}$ are positive.*
(iii) *Both $a\,x^2$ and $(x, y)\mathbf{A}(x, y)^T$ are positive definite.*
(iv) *Both* $\det(\mathbf{A}) > 0$ *and* $a > 0$.
(v) *Both the pivots obtained without row exchanges or scalar multiplications of rows are positive.*
(vi) *By completion of the squares, $\mathbf{Q}(\mathbf{x})$ can be represented as a sum of two squares, with both positive coefficients.*

2. Positive definite quadratic forms

In the general $n \times n$ symmetric case, we will see two conditions similar to these for the 2×2 case. A condition for $\mathbf{Q}$ to be positive definite can be given in terms of several determinants of the "principal" submatrices. Second, $\mathbf{Q}$ is positive definite if the pivots are all positive, and this can be understood in terms of completion of the squares.

Let $\mathbf{A}$ be an $n \times n$ symmetric matrix. We need to consider submatrices of $\mathbf{A}$. Let $\mathbf{A}_k$ be the $k \times k$ submatrix formed by deleting the last $n - k$ rows and last $n - k$ columns of $\mathbf{A}$,

$$\mathbf{A}_k = \left(a_{i,j}\right)_{1 \le i \le k, 1 \le j \le k}.$$

The following theorem gives conditions of the quadratic form being positive definite in terms of determinants of $\mathbf{A}_k$.

Theorem 2. *Let $\mathbf{A}$ be an $n \times n$ symmetric matrix and $\mathbf{Q}(\mathbf{x}) = \mathbf{x}^T \mathbf{A}\mathbf{x}$ the related quadratic form. The following conditions are equivalent:*

(i) $\mathbf{Q}(\mathbf{x})$ *is positive definite.*
(ii) *All the eigenvalues of $\mathbf{A}$ are positive.*
(iii) *For each $1 \le k \le n$, the quadratic form associated to $\mathbf{A}_k$ is positive definite.*
(iv) *The determinants,* $\det(\mathbf{A}_k) > 0$ *for $1 \le k \le n$.*
(v) *All the pivots obtained without row exchanges or scalar multiplications of rows are positive.*
(vi) *By completion of the squares, $\mathbf{Q}(\mathbf{x})$ can be represented as a sum of squares, with all positive coefficients,*

$$\begin{aligned}\mathbf{Q}(x_1, \dots, x_n) &= (x_1, \dots, x_n)\mathbf{U}^T \mathbf{D}\mathbf{U}(x_1, \dots, x_n)^T \\ &= p_1 \left(x_1 + u_{1,2}x_2 + \cdots + u_{1,n}x_n\right)^2 \\ &\quad + p_2 \left(x_2 + u_{2,3}x_3 + \cdots + u_{2,n}x_n\right)^2 \\ &\quad + \cdots + p_n x_n^2.\end{aligned}$$

Proof. We assume $\mathbf{A}$ is symmetric so we can find an orthonormal basis of eigenvectors $\mathbf{v}^1, \dots \mathbf{v}^n$ with eigenvalues $\lambda_1, \dots \lambda_n$. Let $\mathbf{P}$ be the orthogonal matrix formed by putting the $\mathbf{v}^j$ as the columns. Then $\mathbf{P}^T \mathbf{A}\mathbf{P} = \mathbf{D}$ is the diagonal matrix with entries $\lambda_1, \dots \lambda_n$. Setting

$$\mathbf{x} = y_1 \mathbf{v}^1 + \cdots + y_n \mathbf{v}^n = \mathbf{P}\mathbf{y},$$

the quadratic form turns into a sum of squares:

$$\begin{aligned}\mathbf{Q}(\mathbf{x}) &= \mathbf{x}^T \mathbf{A}\mathbf{x} \\ &= \mathbf{y}^T \mathbf{P}^T \mathbf{A}\mathbf{P}\mathbf{y} \\ &= \mathbf{y}^T \mathbf{D}\mathbf{y} \\ &= \sum_{j=1}^{n} \lambda_j \, y_j^2.\end{aligned}$$

From this representation, it is clear that $\mathbf{Q}$ is positive definite if and only if all the eigenvalues are positive, i.e., conditions (i) and (ii) are equivalent.

Assume $\mathbf{Q}$ is positive definite. Then for any $1 \le k \le n$,

$$\begin{aligned}0 &< \mathbf{Q}(x_1, \dots, x_k, 0, \dots, 0) \\ &= (x_1, \dots, x_k, 0, \dots, 0)\mathbf{A}(x_1, \dots, x_k, 0, \dots, 0)^T \\ &= (x_1, \dots, x_k)\mathbf{A}_k(x_1, \dots, x_k)^T\end{aligned}$$

for all $(x_1, \dots, x_k) \ne \mathbf{0}$. This shows that (i) implies (iii).

Assume (iii). Then all all the eigenvalues of $\mathbf{A}_k$ must be positive since (i) and (ii) are equivalent for $\mathbf{A}_k$. Notice that the eigenvalues of $\mathbf{A}_k$ are not necessarily eigenvalues of $\mathbf{A}$. Therefore the determinant of $\mathbf{A}_k$ is positive since it is the product of its eigenvalues. This is true for all k, so this shows that (iii) implies (iv).

Assume (iv). When $\mathbf{A}$ is row reduced, it also row reduces all the $\mathbf{A}_k$ since we do not perform any row exchanges. Therefore the pivots of the $\mathbf{A}_k$ are pivots of $\mathbf{A}$. Also, the determinant of $\mathbf{A}_k$ is the product of the first k pivots, $\det(\mathbf{A}_k) = p_1 \dots p_k$. Therefore

$$p_k = (p_1 \dots p_k)/(p_1 \dots p_{k-1}) = \det(\mathbf{A}_k)/\det(\mathbf{A}_{k-1} > 0,$$

for all k. This proves (v).

Now assume (v). Row reduction can be realized by matrix multiplication on the left by a lower triangular matrix. Therefore, we can write $\mathbf{A} = \mathbf{LDU}$ where $\mathbf{D}$ is the diagonal matrix made up of the pivots, $\mathbf{L}$ is lower triangular with ones on the diagonal, and $\mathbf{U}$ is upper diagonal with ones on the diagonal. Since $\mathbf{A}$ is symmetric, $\mathbf{LDU} = \mathbf{A} = \mathbf{A}^T = \mathbf{U}^T\mathbf{DL}^T$. It can then be shown that $\mathbf{U}^T = \mathbf{L}$. Therefore,

$$\begin{aligned}\mathbf{Q}(x_1, \dots, x_n) &= (x_1, \dots, x_n)\mathbf{U}^T\mathbf{DU}(x_1, \dots, x_n)^T \\ &= p_1\,(x_1 + u_{1,2}x_2 + \dots + u_{1,n}x_n)^2 \\ &\quad + p_2\,(x_2 + u_{2,3}x_3 + \dots + u_{2,n}x_n)^2 \\ &\quad + \dots + p_n x_n^2.\end{aligned}$$

Thus, we can "complete the squares", expressing $\mathbf{Q}$ as the sum of squares with the pivots as the coefficients. If the pivots are all positive, then all the coefficients p_i are positive. Thus (v) implies (vi). Note, that $\mathbf{z} = \mathbf{Ux}$ is a non-orthonormal change of basis that makes the quadratic form diagonal.

If $\mathbf{Q}(\mathbf{x})$ can be written as the sum of squares of the above form with positive coefficients, then the quadratic form must be positive. Thus, (vi) implies (i). □

Example 3. Let

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

The eigenvalues are 2 and $2 \pm \sqrt{2}$ which are all positive, which shows that the quadratic form induced by $\mathbf{A}$ is positive definite. (Notice that these eigenvalues are not especially easy to calculate.)

We can row reduce to represent $\mathbf{A}$ as the product of lower triangular, diagonal, and upper triangular matrices.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the pivots on the diagonal are all positive, the quadratic form induced by $\mathbf{A}$ is positive definite, and

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = 2(x_1 - \frac{1}{2}x_2)^2 + \frac{3}{2}(x_2 - \frac{2}{3}x_3)^2 + \frac{4}{3}x_3^2.$$

The principal submatrices and their determinants are

$$\begin{aligned} &\mathbf{A}_1 = (2), && \det(\mathbf{A}_1) = 2 > 0, \\ &\mathbf{A}_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, && \det(\mathbf{A}_2) = 3 > 0, \\ &\mathbf{A}_3 = \mathbf{A} && \det(\mathbf{A}) = 2\left(3/2\right)\left(4/3\right) = 4 > 0. \end{aligned}$$

Since these are all positive, the quadratic form induced by $\mathbf{A}$ is positive definite.

3. NEGATIVE DEFINITE QUADRATIC FORMS

The conditions for the quadratic form to be negative definite are similar, all the eigenvalues must be negative.

Theorem 4. *Let* $\mathbf{A}$ *be an* $n \times n$ *symmetric matrix and* $\mathbf{Q}(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ *the related quadratic form. The following conditions are equivalent:*

(i) $\mathbf{Q}(\mathbf{x})$ *is negative definite.*
(ii) *All the eigenvalues of* $\mathbf{A}$ *are negative.*
(iii) *The quadratic forms associated to all the* $\mathbf{A}_k$ *are negative definite.*
(iv) *The determinants,* $(\text{-}1)^k \det(\mathbf{A}_k) > 0$ *for* $1 \le k \le n$, *i.e.,* $\det(\mathbf{A}_1) < 0$, $\det(\mathbf{A}_2) > 0, \ldots,$ $(\text{-}1)^n \det(\mathbf{A}_n) = (\text{-}1)^n \det(\mathbf{A}) > 0$.
(v) *All the pivots obtained without row exchanges or scalar multiplications of rows are negative.*
(vi) *By completion of the squares,* $\mathbf{Q}(\mathbf{x})$ *can be represented as a sum of squares, with all negative coefficients,*

$$\begin{aligned}\mathbf{Q}(x_1, \ldots, x_n) &= (x_1, \ldots, x_n)\mathbf{U}^T\mathbf{D}\mathbf{U}(x_1, \ldots, x_n)^T \\ &= p_1\,(x_1 + u_{1,2}x_2 + \cdots + u_{1,n}x_n)^2 \\ &\quad + p_2\,(x_2 + u_{2,3}x_3 + \cdots + u_{2,n}x_n)^2 \\ &\quad + \cdots + p_n x_n^2.\end{aligned}$$

For condition (4), the idea is that the product of k negative numbers has the same sign as $(\text{-}1)^k$.

4. PROBLEMS

1. Decide whether the following matrices are positive definite, negative definite, or neither:

(a) $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{pmatrix}$

REFERENCES

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