M413 - Fall 2013 - Enrique Areyan - <u>HW 1</u> (1

(1) Let x, y, z be Real numbers. Prove that

(a) $(x+y)z = xz + yz$

(b) If $x > y$ then $x+z > y+z$

NP

<u>Pf</u>: (a) As defined in class, let $\mathcal{C}_{\mathbb{Q}} = \{$all rational cauchy sequences$\}$ and the binary relation $\sim$ over $\mathcal{C}_{\mathbb{Q}}$ such that for $c, d \in \mathcal{C}_{\mathbb{Q}}$ $c \sim d$ iff $c_n - d_n \to 0$ as $n \to \infty$. Now, let x, y, z be Real numbers, i.e. $x_n, y_n, z_n \in \mathcal{C}_{\mathbb{Q}}$. We want to prove that $[(x+y)z] = [xz+yz]$ (the equivalence classes are equal). By definition:

$$[(x+y)z] = \{s \in \mathcal{C}_{\mathbb{Q}} \mid s \sim (x+y)z \Leftrightarrow s_n - [(x_n+y_n)z_n] \to 0 \text{ as } n \to \infty\}. \text{ and}$$

$$[xz+yz] = \{s \in \mathcal{C}_{\mathbb{Q}} \mid s \sim (xz+yz) \Leftrightarrow s_n - (x_nz_n + y_nz_n) \to 0 \text{ as } n \to \infty\}.$$

Need to prove double contention: $[(x+y)z] \subseteq [xz+yz]$ and $[xz+yz] \subseteq [(x+y)z]$

($\subseteq$) Let $s \in [(x+y)z]$. By definition $s \sim (x+y)z \Leftrightarrow s_n - [(x_n+y_n)z_n] \to 0$ as $n \to \infty$

$\Leftrightarrow$ Given $\overset{\swarrow\text{rational}}{\varepsilon > 0} : \exists \overset{\swarrow\text{integer}}{N} : |s_n - (x_n+y_n)z_n| < \varepsilon \quad \forall n \geq N$. Let $\varepsilon > 0$ and pick N & $n \geq N$ such that the previous inequality holds. then

$$|s_n - (x_nz_n + y_nz_n)| = |s_n - [(x_n+y_n)z_n]| < \varepsilon \quad \text{by distibution on } \mathbb{Q}$$

therefore, $s_n - (x_nz_n + y_nz_n) \to 0$ as $n \to \infty \Leftrightarrow s \sim (xz+yz) \Leftrightarrow s \in [xz+yz]$.

Hence, $[(x+y)z] \subseteq [xz+yz]$

($\supseteq$) Let $s \in [xz+yz]$. Note that all statements in the proof for ($\subseteq$) are reversible, i.e., the direction ($\Leftarrow$) is also proved. therefore $[xz+yz] \subseteq [(x+y)z]$ which means that $[(x+y)z] = [xz+yz]$ and so, given $x, y, z \in \mathbb{R}$: $(x+y)z = xz+yz$

(b) Let $x > y$. then $x - y > 0$ so that $\exists \overset{\swarrow\text{rational}}{\varepsilon > 0} : \exists \overset{\swarrow\text{integer}}{N} : x_n - y_n \geq \varepsilon \quad \forall n \geq N$.

Pick ε, N and $n \geq N$, such that the above inequality holds. then:

$$\begin{aligned} |x_n - y_n| &= |x_n - y_n + z_n - z_n| \\ &= |(x_n + z_n) - (y_n + z_n)| \\ &\geq \varepsilon \end{aligned}$$

therefore, choosing the same ε and N such that $x > y$. we can conclude that $(x+z) - (y+z) > 0 \Leftrightarrow x+z > y+z$ □