M413 - Enrique Areyan - Fall 2013 - Hw 3

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(1) Let $\{I_n\}$ be a non-increasing sequence of non-empty closed intervals of $\mathbb{R}$. Prove that $\bigcap_n I_n \neq \emptyset$

Pf: Let $I_n = [a_n, b_n]$. By hypothesis $a_n \leq b_n$ and $\begin{cases} a_n \leq a_{n+1} \leq \cdots \\ b_n \geq b_{n+1} \geq \cdots \end{cases}$

Consider the set $S = \{a_n : n \in \mathbb{N}\}$. this set is not empty since by hypothesis $I_n \neq \emptyset$ for any n. Also, this set is bounded above since $b_1 \geq a_n$ for all n. therefore, S has a least upper bound. Let $x = \sup S$.

By properties of being an upper bound we have that $\boxed{x \geq a_n}$ for all n. Moreover, since $b_n \geq a_n$ for all n, b_n is an upper bound. But x is the least upper bound so $\boxed{b_n \geq x}$. Combining these two inequalities we get

$$\boxed{a_n \leq x \leq b_n} \Rightarrow x \in I_n, \text{ for any } n.$$

therefore, $x \in \bigcap_n I_n$ and so $\bigcap_n I_n \neq \emptyset$. □

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(2) Prove that a set A is infinite iff there exists a proper subset B of A such that $B \sim A$.

Pf: ($\Leftarrow$) Suppose that there exists a proper subset B of A s.t. $B \sim A$. Also, suppose for a contradiction that A is finite with $|A| = n$. Since $B \sim A$, we know of the existence of a 1-1, onto function $f: B \to A$. Now, since A is finite and B is a proper subset of A, we can conclude that B is finite. Let $|B| = m \leq n-1$. then, consider the cardinality of the image of f

$$n = |A| \underset{\substack{f \text{ is onto} \\ f \text{ is } 1-1}}{=} |Img(f)| = |\{f(b_1), f(b_2), \ldots, f(b_m)\}| = m \leq n-1$$

Since f is 1-1, the set $\{f(b_1), f(b_2), \ldots, f(b_m)\}$ contains distinct elements ($b_1, b_2 \in B$: $f(b_1) = f(b_2) \Rightarrow b_1 = b_2$, conversely $b_1 \neq b_2 \Rightarrow f(b_1) \neq f(b_2)$). But f is onto and so every element of A has a preimage under f. this justifies our reasoning before but this leads to a clear contradiction

$$n \leq n-1$$

therefore, A is not a finite set, which means that A is infinite (countable or uncountable).

$\Rightarrow$) Suppose that A is infinite.

laim: A contains a countable infinite subset.

Pf (claim): Since A is infinite and not empty, pick $a_0 \in A$ (any element). low that you have a_0, pick another element $a_1 \in A \setminus \{a_0\}$. there is another lement to pick since A is infinite. Proceed inductively by picking the next lement a_k from $A \setminus \{a_0, a_1, \dots, a_{k-1}\}$. then the subset $\{a_0, a_1, \dots, a_k, \dots\}$ is early countable by the mapping $f: \{a_0, a_1, \dots, a_k, \dots\} \to \mathbb{N}$ given by $f(a_i) = i$. (end of claim)

y previous claim, since A is infinite, it has a countable infinite subset. t $A_0 = \{a_0, a_1, a_2, \dots\}$ be such that $A_0 \subset A$. Now, we can prove that A is ui valent to $A \setminus \{a_1, a_3, a_5, \dots\}$, which is clearly a proper subset of A. onsider the 1-1, onto mapping $f: A \to A \setminus \{a_1, a_3, a_5, \dots\}$ given by:

$$f(x) = \begin{cases} a_{2i} & \text{if } x = a_i \text{ ; for } i = 0,1,2,\dots \text{ (this maps } a_0 \to a_0,\ a_1 \to a_2,\ a_2 \to a_4, \dots) \\ x & \text{otherwise (for all other elements use the identity).} \end{cases}$$

is is clearly a one-to-one and onto mapping. this is obvious when we e the identity map. For all other elements f maps integers indices to ven integer indices ($f(a_i) = a_{2i}$), a map which we proved in class to be 1-1 and onto. erefore, f is a bijection from A to $A \setminus \{a_1, a_3, a_5, \dots\}$ which is a proper subset A, showing that $A \sim A \setminus \{a_1, a_3, a_5, \dots\}$.

Exhibit explicit 1-1, onto functions f, g s.t: $f: (0,1) \to [0,1]$ and $g: (0,1) \to \mathbb{R}$.

ution: First, consider the function $g: (0,1) \to \mathbb{R}$ given by

$$\boxed{g(x) = \tan((x + \tfrac{1}{2})\pi)}$$

trigonometric properties of the tangent function, $\tan(x)$ is a 1-1 and to function from $(\frac{-\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$. By taking $\tan((x+\frac{1}{2})\pi)$, we have shifted and led the function to be a bijection from $(0,1)$ to $\mathbb{R}$. An alternative $\mathbb{R} \to (0,1)$ is $g(x) = \frac{\arctan(x)}{\pi} + \frac{1}{2}$, which is also a bijection for similar reasons.

mally, let us show

g is 1-1: Let $x, y \in (0,1)$ be such that $g(x) = g(y)$ $\Leftrightarrow$

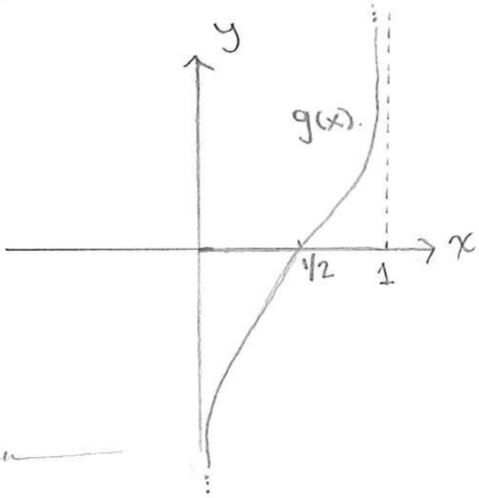
$\tan((x+\frac{1}{2})\pi) = \tan((y+\frac{1}{2})\pi)$; apply arctan to both sides $\Leftrightarrow$

$\arctan(\tan((x+\frac{1}{2})\pi)) = \arctan(\tan((y+\frac{1}{2})\pi)) \Leftrightarrow (x+\frac{1}{2})\pi = (y+\frac{1}{2})\pi \Leftrightarrow x+\frac{1}{2} = y+\frac{1}{2} \Leftrightarrow \boxed{x=y}$

g is onto: Let $y \in \mathbb{R}$. take $x = \frac{\arctan(y)}{\pi} - \frac{1}{2}$. then,

$g(x) = g\left(\frac{\arctan(y)}{\pi} - \frac{1}{2}\right) = \tan\left(\left(\frac{\arctan(y)}{\pi} - \frac{1}{2} + \frac{1}{2}\right)\pi\right) = \tan(\arctan(y)) = y.$

Graphically, g is:



Second, for a function $f:(0,1)\to[0,1]$, consider the following:

Since $[0,1]$ is an infinite set, we know from (2) that it is equivalent to some proper subset $B\subset[0,1]$. In fact, $[0,1]\sim(0,1)$. To prove this claim, use the fact proved in class that $\mathbb{Q}$ is countable. Since $(0,1)\cap\mathbb{Q}\supset\{\frac{1}{n}, n\in\mathbb{N}\}$ is infinite and $(0,1)\cap\mathbb{Q}\subset\mathbb{Q}$, by theorem proved in class we know that $(0,1)\cap\mathbb{Q}$ is countable. therefore, we can list its elements $(0,1)\cap\mathbb{Q}=\{r_1,r_2,r_3,\dots,r_n,\dots\}$

Now consider the mapping $f:[0,1]\to(0,1)$ given by:

$f(0)=r_1$, $f(1)=r_2$, $f(r_1)=r_3$, $f(r_2)=r_4$, $f(r_3)=r_5$, ..., $f(r_i)=r_{i+2}$, $i=1,2,3,\dots$

$f(x)=x$ for all other irrational numbers.

this map is clearly 1-1 and onto, therefore $[0,1]\sim(0,1)$.

this shows the resdt we wanted but in an implicit form. However, we can use this idea to develop the following explicit, 1-1 and onto mapping $f:[0,1]\to(0,1)$

Let $x\in[0,1]$. then,
$$f(x)=\begin{cases}1/2 & \text{if } x=0\\ 1/3 & \text{if } x=1\\ 1/(n+2) & \text{if } x=\frac{1}{n},\ n>1\\ x & \text{otherwise}\end{cases}$$

claim: this is a 1-1, onto mapping. Clearly, when f acts as the identity, the map is 1-1, onto. For other cases:

1-1: take $x,y\in[0,1]$ with $f(x)\neq x$ and $f(y)\neq y$ (nothing to show there).
Suppose $f(x)=f(y)$. then, eithe $x=y=0$ OR $x=y=1$ OR
$f(x)=f(y)\iff f(\frac{1}{n})=f(\frac{1}{m})$, $n,m>1$ $\iff \frac{1}{n+2}=\frac{1}{m+2}\iff n+2=m+2\iff n=m$

Hence, f is 1-1.

nto: Let $y \in (0,1)$. If $y \neq \frac{1}{n+2}$, $n \geq 1$ then take $x = y$ to get $f(x) = f(y) = y$.
therwise, if $y = \frac{1}{n+2}$, $n \geq 1$ take $x = \frac{1}{n}$ then $f(x) = f(\frac{1}{n}) = \frac{1}{n+2} = y$.
ence, f is onto.

ince f is 1-1 and onto where $f: [0,1] \to (0,1)$; f is a bijection and so
e know of the existence of $f^{-1}: (0,1) \to [0,1]$. In this case the inverse
, easily stated as:

$$f^{-1}(x) = \begin{cases} 0 & \text{if } x = 1/2 \\ 1 & \text{if } x = 1/3 \\ \frac{1}{n} & \text{if } x = \frac{1}{n+2}, \ n \geq 1 \\ x & \text{otherwise} \end{cases}$$

nd this is the function we wanted.

he whole point here was to take a countable sequence out of $(0,1)$, shift it
y 2 to make room for 0 and 1, and send all others outside ~~the sequence to~~
hem selves]