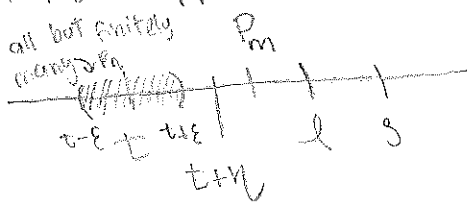
M413 - Fall 2013 - HW7 - Enrique Areyan

(1) Prove that if t is a limit point of a bounded sequence of real numbers $\{p_n\}$, then $\liminf_n p_n \leq t$.

Pf: Let $l = \liminf_n p_n$. Suppose for a contradiction that $l > t$



, where $s = \limsup_n p_n$. We prove in class that: $\liminf_n \{p_n\} \leq \limsup_n \{p_n\}$

Let $\eta = \frac{l-t}{2}$; then, applying proposition given in class to $a = t+\eta$, we have: $a = t+\eta < s \Rightarrow \forall k : \exists m \geq k$, s.t: $p_m > a = t+\eta$. Pick such k and $m \geq k$.

Now, t is a limit point of p_n, so all but finitely many points of $\{p_n\}$ are going to be in any neighborhood of t. But, let $0 < \epsilon < t+\eta$; then the neighborhood of t of radius ϵ does not contain p_m for infinitely many terms, one for each k that you pick. this contradicts the fact that t is a limit point of $\{p_n\}$.

therefore, $l \leq t \Leftrightarrow \boxed{\liminf_n p_n \leq t}$

(2) Prove that in $\mathbb{R}$, every Cauchy sequence converges.

Pf: We prove in class the following Corollary:

$\{p_n\}$ bounded and $p_n \to p$ iff $p = \limsup_n p_n = \liminf_n p_n$

therefore, it suffices to show that for $\{p_n\}$ a Cauchy sequence in $\mathbb{R}$ we have that $\limsup_n p_n = \liminf_n p_n$.

Note that we proved, some time ago, that Cauchy implies boundeness.

Let $\{p_n\}$ be a Cauchy sequence in $\mathbb{R}$. let us prove:

(i) $\liminf_n \{p_n\} \leq \limsup_n \{p_n\}$

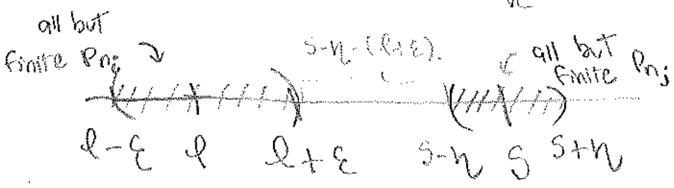
(ii) $\limsup_n \{p_n\} \leq \liminf_n \{p_n\}$

(i) This is a property of lim inf and lim sup, we already prove this.

ii) Let us prove that, given $\varepsilon > 0$ $\limsup_n p_n \le \liminf_n p_n + \varepsilon$

ppose, for a contradiction, that there exists $\varepsilon > 0$ s.t:

$$S = \limsup_n p_n > \liminf_n p_n + \varepsilon = \ell + \varepsilon$$



choose $\eta = \dfrac{S-(\ell+\varepsilon)}{2}$

Now, since $\{p_n\}$ is bounded, we have that

there exists a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ s.t. $p_{n_i} \to \ell$. Likewise, there exists a subsequence $\{p_{n_j}\}$ of $\{p_n\}$ s.t $p_{n_j} \to S$.

ence, all but finitely many points of $\{p_{n_i}\}$ are going to be in the neighborhood ; ℓ of radius ε. Likewise, all but finitely many points of $\{p_{n_j}\}$ are ing to be in the neighborhood of S of radius η.

wever, $\{p_n\}$ is Cauchy, which means that $\forall r > 0 : \exists N : \forall_{k,m} \ge N : |p_k - p_m| < r$. for $r = S - \eta - (\ell+\varepsilon)$ and for any N, we can always find $n_i, n_j \ge N$ s.t. $|p_{n_i} - p_{n_j}| \ge r$; where p_{n_i} converges to ℓ and p_{n_j} converges to S as described ove; just pick n_i, n_j large enough. But this contradicts the fact that is Cauchy. therefore $\limsup_n p_n \le \liminf_n p_n + \varepsilon$, for any ε

$\limsup_n p_n \le \liminf_n p_n$.

$\S$ (ii) $\Rightarrow$ $\limsup_n p_n = \liminf_n p_n$, call them p. By previous theorem,

$p_n \to p$. since p_n was an arbitrary Cauchy sequence we obtain the result

Let $p_n = \{p_1, p_2, \ldots\}$ be a sequence of all rational numbers in $[0,1]$.

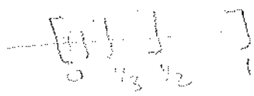
Find (i) $\liminf_n p_n$ and (ii) $\limsup_n p_n$

tion: A theorem proved in class states that for a bounded sequence eal numbers and $t \in \mathbb{R}$; if $\{p_{n_k}\}$ (a subsequence) converges to t, then

$$\liminf_n p_n \le t \le \limsup_n p_n$$

e p_n is a bounded, non-empty set of real numbers, it has a sup, and is that $1 = \sup p_n$ and $0 = \inf p_n$. Hence $0 \le \liminf p_n \le \limsup p_n \le 1$

therefore, if we can find a subsequence of $\{p_n\}$ that converges to 0, we can conclude that $0 \leq \liminf_n p_n \leq 0 \Rightarrow \liminf_n p_n = 0$. Likewise, if we can find a subsequence of $\{p_n\}$ that converges to 1, we can conclude that $1 \leq \limsup_n p_n \leq 1 \Rightarrow \limsup_n p_n = 0$.



We can indeed find such subsequences:

(i) the following is a subsequence of p_n that converges to 0:

p_{n_1}; pick any rational in p_n s.t $0 < p_n < 1$. Then $p_{n_1} = p_n$

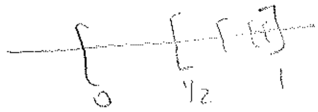
p_{n_2}; pick any rational enumerated after p_{n_1} in $\{p_n\}$ s.t. $0 < p_{n_2} < 1/2$; $p_{n_2} \neq$

p_{n_3}; pick any rational enumerated after p_{n_2} in $\{p_n\}$ s.t. $0 < p_{n_3} < 1/3$; $p_{n_3} \neq$

Having picked $p_{n_{k-1}}$; pick p_{n_k} any rational enumerated after $p_{n_{k-1}}$ in $\{p_n\}$ s.t. $0 < p_{n_k} <$ p_{n_k}

this is a subsequence of p_n converging to 0. therefore, $\boxed{0 = \liminf_n p_n}$

(ii) the following subsequence of p_n converges to 1:



$p_{n_l} = 1 - p_{n_k}$; with p_{n_k} picked as in (i).

then p_{n_l} is a subsequence of p_n that converges to 1.

therefore, $\boxed{1 = \limsup_n p_n}$

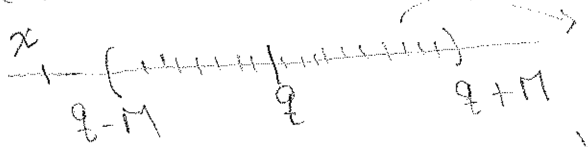
(4) Given a bounded sequence $\{p_n\}$, define $X = \{p \in \mathbb{R} :$ infinitely many $p_n > p$

Prove the following:

(a) X is bounded above.

(b) $s = \sup X = \limsup_n p_n$

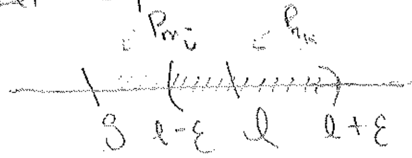
Pf (a) By definition, $\{p_n\}$ is bounded if its range is bounded. Hence, there exists a real number M and $q \in \mathbb{R}$ s.t. $|p_n - q| < M$ for all p_n.



Clearly $q+M > p_n > p \Rightarrow q+M$ is an upper bound of X. Moreover, $X \neq$ since any point $x \leq q-M$ is going to be less than p_n for infinitely many (in fact for all), $x = q-M \in X$. So, X is not empty.

b) By part ⓐ, since $X \neq \emptyset$ and X is bounded above, X has a sup.
Let $s = \sup X$. Claim $s = \limsup_n \{p_n\} = l$. Pf: Let us prove

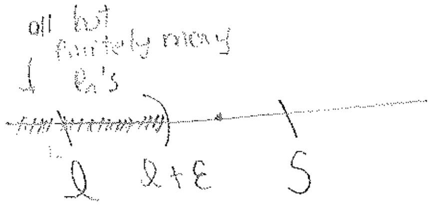
ⓐ $s \geq \limsup_n \{p_n\} = l$



Suppose for a contradiction that $s < l$.
Consider the following two properties:
(i) there exists $\{p_{n_k}\}$ s.t. $p_{n_k} \to l$. For $\varepsilon = \frac{l-s}{2}$, there exists N s.t.
all points p_{n_k}, for $n_k \geq N$; $p_{n_k} > l - \varepsilon$. Hence, $l - \varepsilon \in X$.
(ii) since $s < l \Rightarrow$ for all k, there exist $m \geq k$ s.t. $p_m > s$. Consider
$k = 1, 2, 3, \dots$; choose $m_1, m_2, m_3, \dots$ s.t. $p_{m_i} > s$. Then, $s \in X$.
(i) & (ii) $\Rightarrow$ $s < l - \varepsilon$, but $l - \varepsilon \in X$ so $s \geq l - \varepsilon$; a contradiction.

Hence, $s \geq l$.

ⓑ $s \leq \limsup_n \{p_n\} = l$.



Suppose for a contradiction that $s > l$.
Consider the following two properties:
i) Let $\varepsilon = \frac{s-l}{2} > 0$. Since $l + \varepsilon > l \Rightarrow$ there exists K s.t. $\forall n \geq K$ $p_n < l+\varepsilon$. Therefore, $l+\varepsilon \in X$.
all but finitely many points are to the left of $l+\varepsilon$.
ii) ALL points between $l+\varepsilon$ and s are not in X, since there are only finitely many p_n's between $l+\varepsilon$ and s. This follows from (i). Moreover,
$l+\varepsilon$ is an upper bound for X
(i) & (ii) $\Rightarrow$ $l + \varepsilon > x$ $\forall x \in X$ and $s > l+\varepsilon$. But s is the l.u.b of X
$s \leq l + \varepsilon$; a contradiction. Hence $s \leq l$.
Both $s \geq l$ and $s \leq l \Rightarrow$ $s = l \iff \left[s = \limsup_n \{p_n\} \right]$

(5) (a) Compare $\limsup_n (p_n+p_n')$ with $\limsup_n \{p_n\} + \limsup_n \{p_n'\}$.
(b) If $\lim_n p_n = p$ then $\limsup_n (p_n+p_n') = \limsup_n \{p_n\} + \limsup_n \{p_n'\}$

Solution
(a) Claim $\limsup_n (p_n+p_n') \leq \limsup_n \{p_n\} + \limsup_n \{p_n'\}$.

An example is $p_n = 0,1,0,1,\ldots$; $p_n' = 1,0,1,0,\ldots \Rightarrow p_n+p_n' = 1,1,1,1,\ldots$
$\limsup_n (p_n+p_n') = 1 \leq 2 = 1+1 = \limsup_n \{p_n\} + \limsup_n \{p_n'\}$.

Pf. By definition of $\limsup_n (p_n+p_n')$, $\exists$ a subsequence $\{p_{n_k}+p_{n_k}'\}$ s.t

$$\lim_n (p_{n_k}+p_{n_k}') = \limsup_n (p_n+p_n').$$

$\{p_{n_k}\}$ is a subsequence of $\{p_n\}$, hence it is bounded. So there exists a subsequen of $\{p_{n_k}\}$, say $\{p_{n_{k_\ell}}\}$ that converges.
However, $\{p_{n_{k_\ell}}'\}$ may or may not converge, but we know it is bounded. So there exists a subsequence of $\{p_{n_{k_\ell}}'\}$ say $\{p_{n_{k_{\ell_m}}}'\}$ that converges.
Since $\{p_{n_{k_\ell}}\}$ converges, so does any subsequence of it, in particular $\{p_{n_{k_{\ell_m}}}\}$.
So we have two convergent sequences: $\{p_{n_{k_{\ell_m}}}\}$ and $\{p_{n_{k_{\ell_m}}}'\}$. We proved that the sum of convergent, real valued sequences also converges.
Hence, $\{p_{n_{k_{\ell_m}}} + p_{n_{k_{\ell_m}}}'\}$ also converges. But $\{p_{n_{k_{\ell_m}}} + p_{n_{k_{\ell_m}}}'\}$ is a subseq of $\{p_{n_k}+p_{n_k}'\}$, hence it converges to the same value, which we assume is $\limsup_n (p_n+p_n')$. Thus, $\lim_n \{p_{n_{k_{\ell_m}}} + p_{n_{k_{\ell_m}}}'\} = \limsup_n \{p_n+p_n'\}$

But then, $\lim_n \{p_{n_{k_{\ell_m}}} + p_{n_{k_{\ell_m}}}'\} = \lim_n \{p_{n_{k_{\ell_m}}}\} + \lim_n \{p_{n_{k_{\ell_m}}}'\} = \limsup_n \{p_n+p_n'\}$
Finally, $\lim_n \{p_{n_{k_{\ell_m}}}\}$ is a subsequential limit point of $\{p_n\}$. therefore,

$$\lim_n \{p_{n_{k_{\ell_m}}}\} \leq \limsup_n \{p_n\}. \text{ Likewise, } \lim_n \{p_{n_{k_{\ell_m}}}'\} \leq \limsup_n \{p_n'\}$$

Combining these:

$$\limsup_n \{p_n+p_n'\} = \lim_n \{p_{n_{k_{\ell_m}}}\} + \lim_n \{p_{n_{k_{\ell_m}}}'\} \leq \limsup_n \{p_n\} + \limsup_n \{p_n'\}$$ //

b). By part ⓐ, we need only to prove:

$$\limsup_n \{p_n + p_n'\} \geq \limsup_n p_n + \limsup_n p_n' ,$$ provided that

$\lim_n p_n = p$, exists.

et $\{p'_{n_k}\}$ be a subsequence of $\{p_n'\}$ that converges to $\limsup_n \{p_n'\}$, i.e.,

$$\lim_{n_k} (\{p'_{n_k}\}) = \limsup_n \{p_n'\}.$$

ow, since we are assuming that $\{p_n\}$ converges, any subsequence of will converge to the same value, In particular $\{p_{n_k}\}$. therefore:

$$\lim_{n_k} \{p_{n_k}\} = p = \limsup_n \{p_n\} .$$

us, we can add convergent sequences and obtain a convergent sequence:

$$\lim_{n_k}\{p_{n_k}\} + \lim_{n_k}\{p'_{n_k}\} = \lim_{n_k}\{p_{n_k} + p'_{n_k}\} = \limsup_n \{p_n\} + \limsup_n \{p_n'\}$$

But; $\lim_{n_k} \{p_{n_k} + p'_{n_k}\} \leq \limsup_n \{p_n + p_n'\}$. thus,

$$\boxed{\limsup_n \{p_n + p_n'\} \geq \limsup_n \{p_n\} + \limsup_n \{p_n'\}}$$

ghter with part ⓐ, this implies that

$\limsup_n \{p_n + p_n'\} = \limsup_n \{p_n\} + \limsup_n \{p_n'\}$, Provided that $p_n \to p$.