M413 - Fall 2013 - Enrique Areyan - Exam 1.

Problem 1:

(b) Let $F(\mathbb{N}) = \{A : A \text{ is finite}, A \subset \mathbb{N}\}$. claim: $F(\mathbb{N})$ is countable.

Pf: to prove this I will use the fact proved in class that the union of countable sets is countable. thus, by showing that $F_n(\mathbb{N}) = \{A : A \text{ is finite}, A \subset \mathbb{N}, |A| \leq n\}$ is countable for each n and letting $n = 1, 2, 3, \ldots$; We will show that $F(\mathbb{N}) = \bigcup_{n=1}^{\infty} F_n(\mathbb{N})$ is countable

Consider the function:

This function doesn't look good, it is not 1-1.

$f : F_n(\mathbb{N}) \to \mathbb{N}$, given by $f(\{a_1, \ldots, a_n\}) = n$; this function is onto a subset of $\mathbb{N}$; that is onto $\{1, \ldots, n\}$. therefore, by result proved in class $F_n(\mathbb{N}) \subset$ subset of $\mathbb{N}$, $F_n(\mathbb{N})$ infinite and $\mathbb{N}$ countable $\Rightarrow F_n(\mathbb{N})$ is at most countable. Hence, by letting $n = 1, 2, \ldots$ as explained before, we find that $F(\mathbb{N})$ is countable.

Problem 2 : Let us prove (i) $d_1(p,q)$ is a metric on X and
(ii) d_1 is equivalent to d.

i) d_1 satisfies the following three properties:

1. Let $p \in X$. then $d_1(p,p) = \min\{d(p,p), 1\}$ by definition of d_1.
$= \min\{0, 1\}$, since d is a metric, $d(x,x)=0$ $\forall x \in X$.
$= 0$

2. Let $p \in X$ and $q \in X$. then.
$d_1(p,q) = \min\{d(p,q), 1\}$ by definition of d_1
$= \min\{d(q,p), 1\}$ since d is a metric: $d(x,y) = d(y,x)$, $\forall x,y \in X$.
$= d_1(q,p)$.

Moreover, if ~~If~~ $p \neq q$, then $d_1(p,q) = \min\{d(p,q), 1\}$
$= \min\{a, 1\}$, where $a > 0$ since d is a metric
so $d(p,q) > 0$, provided that $p \neq q$. > 0.

3. Let $p,q,r \in X$. then,
$d_1(p,q) = \min\{d(p,q), 1\}$; $d_1(p,r) = \min\{d(p,r), 1\}$; $d_1(r,q) = \min\{d(r,q), 1\}$. by def of d_1.
$d_1(p,r) + d_1(r,q) = \min\{d(p,r), 1\} + \min\{d(r,q), 1\}$.
$\geq \min\{d(p,q), 1\}$, since d is a metric, so $d(p,q) \leq d(p,r) + d(r,q)$
$= d_1(p,q)$ by def of d_1.

By 1, 2 and 3 we conclude that d_1 is a metric.

ii) Want to prove $\lim_n d(p_n, p) = 0 \overset{?}{\iff} \lim_n d_1(p_n, p) = 0$.

(⇒) Suppose $\lim_n d(p_n,p) = 0$. then, given $\varepsilon > 0$; $\exists N$ s.t. $\forall n \geq N$ $d(d(p_n,p),0) < \varepsilon$. ?
Want to show that Given $\varepsilon' > 0$ $\exists M$ s.t $\forall m \geq M$: $d_1(d_1(p_n,p),0) < \varepsilon'$.
Let $\varepsilon > 0$ and pick M s.t. $d(d(p_n,p),0) < \varepsilon$, whenever $n \geq M$. then
$d_1(d_1(p_n,p),0) = d_1(\min\{d(p_n,p),1\}, 0)$ by definition of d_1
$= \min(\min\{d(p_n,p),1\}, 0)$ by definition of d_1.
$= 0$, since d is a metric and hence always positive
$< \varepsilon$.

so that $\lim_n d_1(p_n,p) = 0$, provided that $\lim_n d(p_n,p) = 0$

($\Leftarrow$) Suppose $\lim_n d_1(p_n,p)=0$. then, given $\varepsilon>0$ $\exists N: \forall n\geq N: d_1(d_1(p_n,p),0)<\varepsilon$

I want to show that Given $\varepsilon'>0: \exists n$ s.t. $\forall m\geq n: d(d(p_n,p),0)<\varepsilon'$.

Let $\varepsilon>0$. Pick N s.t. $d_1(d_1(p_n,p),0)<\varepsilon$ whenever $n\geq N$. then,

$$d_1(d_1(p_n,p),0) = d_1(\min\{d(p_n,p),1\},0)$$
$$=\min\{d(\min\{d(p_n,p),1\},0),1\}<\varepsilon \quad \text{by hypothesis}$$

$\Rightarrow d(\min\{d(p_n,n),1\},0)<\varepsilon$; since this is the only way that the min fraction can be less than ε.

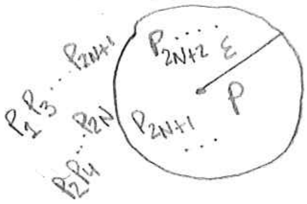
$\Rightarrow \min\{d(p_n,n),1\}<\varepsilon$, same as before,

$\Rightarrow d(p_n,n)<\varepsilon$, otherwise $d(p_n,n)>\varepsilon \Rightarrow \min\{d(p_n,p),1\}>\varepsilon$, not possib

therefore, $\lim_n d(p_n,p)=0$ provided that $\lim_n d_1(p_n,p)=0$.

Problem 3:

(a) Suppose that given a sequence $\{p_n\}$, we have that $\{p_{2n}\}\to p$ and $\{p_{2n+1}\}\to p$. We want to prove that $\{p_n\}\overset{?}{\to}p$.



Neighborhood of p of radius $\varepsilon>0$ all but finitely many p_{2n} and p_{2n+1} are contained in any such neighboorhood.

We can see that a relabeling of this terms immediately imply $p_n\to 0$ as $\{p_n\}=\{p_{2n}\}\cup\{p_{2n+1}\}$. Suppose for a contradiction that p_n does not converge to then p is not a limit point of $\{p_n\}$. that means that there exists $N_r(p)\setminus\{p\}$ that contains finitely many p_n, for some $r>0$. However, since $\{p_{2n}\}\to p$ and $\{p_{2n+1}\}\to p$; every neighborhood of p contains infinitely many p_{2n}'s and p_{2n+1}'s ; in particular the neighborhood of radius r as choose befor for a choice of N. Pick such N, then $p_{2N}, p_{2N+1}, p_{2N+2}, \ldots$, are infinitely many points of p_n (relabeling $n=2N$) contained in $N_r(p)\setminus\{p\}$, a contradiction. th there exists no such neighboorhood $\Rightarrow p$ is a limit point of $\{p_n\}$.

Problem 4:

a) Consider the following real sequences:

$q_n = \frac{1}{n}+1$; $n \in \mathbb{N}$ and $P_n = -1, 1, -1, 1, \ldots$

Then, q_n is convergent. In fact it converges to ~~one~~ 1 since given $\varepsilon > 0$

pick $\frac{1}{N} - 1 < \varepsilon$, then, if $n \geq N$ $\frac{1}{n} - 1 \leq \frac{1}{N} - 1 \Rightarrow \frac{1}{n} - 1 < \varepsilon$

* Note that we can pick such a N by the archimedian property

Also, P_n is bounded. In fact $P_n \subset [-1, 1]$.

However, $\{t_n\} = \{P_n + q_n\} = \{P_n + \frac{1}{n}\}$ is divergent since the subsequence $\{t_{2n+1}\} = \{1 + \frac{1}{n}\} = \{\frac{n+1}{n}\}$; which diverges so the original sequence $\{t_n\}$ diverges.

Finally, $\{s_n\} = \{P_n q_n\} = \begin{cases} \frac{1}{n} + 1 & \text{if } n \text{ is even} \\ \frac{1}{n} & \text{if } n \text{ is odd.} \end{cases}$

One of these subsequences converges to 1 (which was proved before) and the other $\frac{1}{n} \to 0$. Hence, $\{s_n\}$ diverges ✓

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Problem 5: (a) Let $K_1, \ldots, K_n$ be a finite collection of compact subsets.

Then, Given an open cover $\{G_\alpha\}$ of $\bigcup_{i=1}^{n} K_i$, i.e., $\bigcup_{i=1}^{n} K_i \subset \{G_\alpha\}$, where G_α is an open set, we can always obtain a finite subcover as follow:

Since K_i is compact, it has a finite cover say $\{G_{\beta_{ij}}\}$ where $j = 1, \ldots, m$ so that $K_i \in \bigcup_{j=1}^{m} G_{\beta_{ij}}$. Take the union of these open covers for each i to produce $\bigcup_{i=1}^{n} \left(\bigcup_{j=1}^{m} G_{\beta_{ij}} \right)$; a finite open cover of $K_1, \ldots, K_n$ (since the union of finite sets is finite). But then $\bigcup_{i=1}^{n} \bigcup_{j=1}^{m} G_{\beta_{ij}} \subset \{G_\alpha\}$; since otherwise there would be a K_i not completely cover by $\bigcup_{i=1}^{n} \bigcup_{j=1}^{m} G_{\beta_{ij}}$.

This result does not extend to infinite unions: $A_n = [1 - \frac{1}{n}, 2]$, $n = 1, 2, 3, \ldots$

Each set is bounded and close in $\mathbb{R}$ so A_n is compact for a given n, but

$\bigcup_{n \in \mathbb{N}} [1 - \frac{1}{n}, 2] = (0, 2]$ $[0,2]$, which is not closed, hence not compact.

↳ if $n=1$ is included