S413 - SECOND TEST

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If a problem has two parts, do ONLY one; if you do both you will get the lower grade. And, write legibly. I will only grade what I can read.

Problem 1. Part (a) A Buddhist monk leaves his monastery at 7 a.m. and climbs the neighbouring mountain, arriving at the top at 7 p.m. that evening. After a night of meditation he starts descending at 7 a.m. the next day and arrives at his monastery at 7 p.m. that evening. Prove that there is a time t such that the monk was at the same elevation on both days.

Part (b) Let f be a continuous function on $[0,2]$ such that $f(0) = f(2)$. Prove that there exist x_1 and x_2 in $[0,2]$ such that $x_1 = 1 + x_2$ and $f(x_1) = f(x_2)$.

Problem 2. Part (a) Let $f : [a, \infty) \to \mathbb{R}$ be a continuous function. Prove that if the graph of f has a horizontal asymptote at ∞, i.e., $\lim_{x\to\infty} f(x)$ exists, then f is uniformly continuous on $[a, \infty)$.

Part (b) Let $f, g : [a, \infty) \to \mathbb{R}$ be continuous functions such that

$$\lim_{x\to\infty} (f(x) - g(x)) = 0\,.$$

Prove that f is uniformly continuous on $[a, \infty)$ if and only if g is uniformly continuous on $[a, \infty)$.

Problem 3. Part (a) For what values of $\gamma > 0$ it follows that

$$\lim_{n\to\infty} n^\gamma \left(\sqrt[n]{n} - 1\right)^2 = 0\,.$$

Part (b) Give an example of two convergent series $\sum_{n=1}^\infty x_n$ and $\sum_{n=1}^\infty y_n$ such that $\sum_{n=1}^\infty x_n y_n$ diverges. Can this happen if one of the series is absolutely convergent?

Problem 4. Part (a) Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ denote that unit circle in $\mathbb{R}^2$. Prove that S^1 is connected and that if $f : S^1 \to \mathbb{R}$ is continuous, then f satisfies the intermediate value property.

Part (b) Let (X, d) be a metric space and $f : X \to X$ a continuous function. Define a function $g : X \to \mathbb{R}$ by

$$g(p) = d(p, f(p)).$$

Prove that g is continuous.

Problem 5. Part (a) Let (X, d) be a metric space and A, B disjoint closed subsets of X. Prove that there exists a continuous function $f : X \to [0, 1]$ such that $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$.

Hint: Consider the function

$$f(p) = \frac{d(p, A)}{d(p, A) + d(p, B)}.$$

Part (b) Let (X, d) be a metric space and $f : X \to X$ a function that satisfies the following property: For every $p \in X$ there exists a sequence $\{p_n\} \subset X$ such that $p_n \to p$ and $f(p_n) \to f(p)$. Is f necessarily continuous?