The Cauchy Criterion

Definition. We say that (s_n) is a Cauchy sequence if for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n,\ m$ satisfying $n > N,\ m > N$ the following inequality holds:

$$|\, s_n - s_m \,| < \varepsilon. \tag{1}$$

Remak. Here N depends on ε, of course.

Theorem 0.1
(i) Every converging sequence is a Cauchy sequence.
(ii) Every Cauchy sequence converges.

Proof. (i) This part is easy. Suppose that $s = \lim_{n\to\infty} s_n$. Then, for a given $\varepsilon > 0$, we can find an N s. t. $\forall\, n > N$, $|s - s_n| < \frac{\varepsilon}{2}$. If now $m > N$, then also $|s - s_m| < \frac{\varepsilon}{2}$. But then

$$|\, s_n - s_m \,| = |(\, s_n - s) + (s - s_m)\,| \le |\, s - s_n \,| + |\, s - s_m \,| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \ \square$$

(ii) This statement part is deeper. It is convenient to present its proof as consisting of three steps.
1. Note first that if (s_n) is a Cauchy sequence, then it is bounded. To see this, set $\varepsilon = 1$ and choose N so that $\forall n, m > N$ we have: $|\, s_n - s_m \,| < 1$. (At this point we use the fact that (s_n) is a Cauchy sequence.) Fix any $m_0 > N$; the last inequality now implies that for all $n > N$ the following is true:

$$s_{m_0} - 1 < s_n < s_{m_0} + 1.$$

But then for all $n \ge 1$

$$\min\{s_1, ..., s_{m_0-1}, s_{m_0} - 1\} \le s_n \le \max\{s_1, ..., s_{m_0-1}, s_{m_0} + 1\}$$

which proves our statement.
2. We are now in a position to apply to the sequence (s_n) the Bolzano-Weierstrass Theorem. Namely, since (s_n) is bounded, it has a converging subsequence. Denote this subsequence $s_{n_k},\ k = 1, 2, ...$ and let $s = \lim_{k\to\infty} s_{n_k}$.
3. It remains to show that in fact the whole sequence converges to s. To do that, fix an $\varepsilon > 0$ and choose N_1 so that $\forall\, n, m > N_1$ we have: $|\, s_n - s_m \,| < \frac{\varepsilon}{2}$ (we use here the Cauchy property of (s_n)).
Choose N_2 so that $\forall\, n_k > N_2$ and such that s_{n_k} belongs to the subsequence we have: $|\, s - s_{n_k} \,| < \frac{\varepsilon}{2}$ (we use here that $s = \lim_{k\to\infty} s_{n_k}$).
Set $N = \max(N_1, N_2)$ and fix $n_k > N$ and such that s_{n_k} belongs to the subsequence. Then for any $n > N$ we have

$$|\, s - s_n \,| = |\,(s - s_{n_k}) + (s_{n_k} - s_n)\,| \le |\, s_{n_k} - s \,| + |\, s_n - s_{n_k} \,| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \ \square$$