Proofs that Sets are Open

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the steps

A set U is a space X is said to be open if every point $p \in U$ has a radius $\epsilon_p > 0$ about it so that $B_p(\epsilon_p) \subset U$.

Recall that $B_p(\epsilon) = \{x \in X : d(x,p) < \epsilon\}$ and recall that the distance, $d(x,y)$ between points $x, y \in X$ has three properties:

$$i) \qquad d(x,y) = 0 \text{ iff } x = y$$

$$ii) \qquad d(x,y) = d(y,x) \qquad \forall x, y, \in X$$

$$iii)\ triangleinequality \;\; d(x,y) + d(y,z) \geq d(x,z)\ \forall x, y, z \in X$$

These properties can be useful when completing a proof that a set is open.

To show that a set U is open we need to show that given any point p in U we can explicitly find a radius $\epsilon_p > 0$ such that $B_p(\epsilon_p) \subset U$.

The steps of the proof that a given set U is open:

1) Let $p \in U$. Write something about p that you know because it is in U using information about U.

2) Find an $\epsilon_p > 0$ using a picture that you believe will work so that $B_p(\epsilon_p) \subset U$. Then write: "Set $\epsilon_p =$ and fill in an explicit formula for ϵ_p.

3) Write :"I claim that $B_p(\epsilon_p) \subset U$".

To complete step 3 you need to provide a proof of the claim. So write: "Proof of the Claim: Given $x \in B_p(\epsilon_p)$ Show $x \in U$.

You can now prepare to write this proof of the claim using the following first two steps:

1) Let $x \in B_p(\epsilon_p)$.

2) So $d(x,p) < \epsilon_p <$ and fill in the formula for ϵ_p from your second step above.

You then prepare to continue your proof by setting up the last two steps (put these at the bottom of the page):

final) So $x \in U$.

step before final step) a formula which will guarantee that x lies in U.

You then proceed to fill in the proof using the formula in step 2 and the properties of the distance function and algebra and anything you can think of that might lead you to conclude the step before the final step.

Once this is done you have completed the proof of the claim which finishes step three that $B_p(\epsilon_p) \subset U$.

*** An Example: Prove that an open ball in X is an open subset of X. So we want to show that a ball $B_x(r) = \{x : d(x,r) < r\}$ is an open set.

1) Let $y \in U = B_x(r)$.

2) Let $\epsilon_y = r - d(x,y)$ (draw a picture to see why this might work).

3) We claim that $B_y(\epsilon_y) \subset U$.

Proof of the Claim: Given: $z \in B_y(\epsilon_y)$ Show: $z \in U = B_x(r)$.

1) Let $z \in B_y(\epsilon_y)$.

2) $d(z,y) < \epsilon_y = r - d(x,y)$.

3)

4)

5)

6) $z \in U = B_x(r)$.

Now fill in step (5) with a formula describing U, which in this case is $d(x,z) < r$.

Now fill in steps 3-4 to try to get from step 2 to step 5.

This can be done, for example, by saying in step 3 that "$d(x,z) \leq d(x,y) + d(y,z)$ by the triangle inequality" and in step 4 writing "$d(x,z) < d(x,y) + \epsilon_y = d(x,y) + r - d(x,y) = r$ by steps 2 and 3".

This completes the proof of the claim and the proof that the set is open.

To show a set S is not open, the proof is quite different. You need to find a point in the space (just one point p is necessary) and then show that no matter how small a radius you choose for ϵ, the ball $B_p(\epsilon)$ is not inside S.

Example: Show $S = [2,5) \times (1,7)$ is not open. This is a rectangle which has left edge inside the set. That is points like $p = (2,4)$ lie in the set because $2 \in [2,5)$ and $4 \in (1,7)$. Draw this! Draw a ball about this point and see how it doesn't fit inside the rectangle. We need to show that for any $\epsilon > 0$, $B_p(\epsilon)$ is not inside S. That is we can find a point q_ϵ in $B_p(\epsilon)$ which is not in S. In fact we can choose $q_\epsilon = (2 - \frac{\epsilon}{2}, 4)$. Notice $d(q_\epsilon, p) = \epsilon/2$ so $q_\epsilon \in B_p(\epsilon)$. Note also that $q_\epsilon \notin S$ because its first coordinate is less than 2. So we are done.