M436 - Introduction to Geometries - Homework 1

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(Ex. 1) Consider the four points $p_1 = (5,8), p_2 = (-1,-1), q_1 = (-2,-1), q_2 = (3,4)$. Determine the coordinates of the intersection of the line p_1p_2 with the line q_1q_2.

Solution: Let us define two lines $\vec{l}_1$ and $\vec{l}_2$ to be the lines through the points $p_1 = (5,8)$ and $p_2 = (-1,-1)$; and $q_1 = (-2,-1)$ and $q_2 = (3,4)$ respectively. Then,

$$\vec{l}_1(s) := \begin{pmatrix}5\\8\end{pmatrix} + s\left[\begin{pmatrix}-1\\-1\end{pmatrix} - \begin{pmatrix}5\\8\end{pmatrix}\right] = \begin{pmatrix}5\\8\end{pmatrix} + s\begin{pmatrix}-6\\-9\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

$$\vec{l}_2(t) := \begin{pmatrix}-2\\-1\end{pmatrix} + t\left[\begin{pmatrix}3\\4\end{pmatrix} - \begin{pmatrix}-2\\-1\end{pmatrix}\right] = \begin{pmatrix}-2\\-1\end{pmatrix} + t\begin{pmatrix}5\\5\end{pmatrix}, \text{ where } t \in \mathbb{R}$$

To find the intersection set these equal, i.e., $\vec{l}_1 = \vec{l}_2$, which means:

$$\begin{pmatrix}5\\8\end{pmatrix} + s\begin{pmatrix}-6\\-9\end{pmatrix} = \begin{pmatrix}-2\\-1\end{pmatrix} + t\begin{pmatrix}5\\5\end{pmatrix}, \quad \text{from which it follows:}$$

$$\begin{matrix}5-6s=-2+5t\\8-9s=-1+5t\end{matrix} \implies \begin{matrix}7-6s=5t\\9-9s=5t\end{matrix} \implies 7-6s=9-9s \implies 3s=2$$

Therefore, $s = \dfrac{2}{3}$ is the parameter for line $\vec{l}_1$ for which it intersects with $\vec{l}_2$. The coordinates of the intersection are:

$$\vec{l}_1(2/3) = \begin{pmatrix}5\\8\end{pmatrix} + \frac{2}{3}\begin{pmatrix}-6\\-9\end{pmatrix} = \begin{pmatrix}5\\8\end{pmatrix} + \begin{pmatrix}-4\\-6\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

So line $\vec{l}_1$ intersects line $\vec{l}_2$ at the point $\boxed{(1,2)}$

(Ex. 2) Consider the following four points:

$$p_1 = (2,3) \quad p_2 = (3,4) \quad p_3 = (4,5) \quad p_4 = (5,6)$$

and determine which of the points p_i is incident with any of the six lines p_jp_k for $j \neq k$.

Solution: First, note that p_1p_2 is given by the equation $y = x+1$. A simple calculation shows that this is true:

$$\overrightarrow{p_1p_2}(s) := \begin{pmatrix}2\\3\end{pmatrix} + s\left[\begin{pmatrix}3\\4\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix}\right] = \begin{pmatrix}2\\3\end{pmatrix} + s\begin{pmatrix}1\\1\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

Solving this vector equation in terms of x and y we get:

$$\binom{2}{3} + s\binom{1}{1} = \binom{x}{y} \implies \begin{matrix}2+s=x\\3+s=y\end{matrix} \implies \begin{matrix}s=x-2\\3+x-2=y\end{matrix} \implies \boxed{y=x+1}$$

Further note that all points p_i for $i = 1,2,3,4$, satisfy this equation. Therefore, all points lie in the same line, so no matter which two points we choose, all pairs of points define the line $y = x+1$. So all six lines are in fact the same line $y = x+1$. This is true because in Euclidean geometry there is only one line through 2 points in the plane. Hence, all points are incident with all six lines, which is in fact the same line $y = x+1$.

(Ex. 3) Consider the three points $p_1 = (1,0), p_2 = (2,0), p_3 = (3,0)$ and the three points $q_1 = (0,1), q_2 = (0,3), q_3 = (0,5)$. Denote the intersection of the line p_iq_j with the line p_jq_i by r_{ij}. Show that the three points r_{12}, r_{13}, r_{23} are collinear.

Solution: First, let us find the coordinates of r_{12}, r_{13} and r_{23}.

For r_{12}: We need the intersection of p_1q_2 and p_2q_1:

$$\overrightarrow{p_1q_2}(s) := \begin{pmatrix}1\\0\end{pmatrix} + s\left[\begin{pmatrix}0\\3\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\right] = \begin{pmatrix}1\\0\end{pmatrix} + s\begin{pmatrix}-1\\3\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

$$\overrightarrow{p_2q_1}(t) := \begin{pmatrix}2\\0\end{pmatrix} + t\left[\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}2\\0\end{pmatrix}\right] = \begin{pmatrix}2\\0\end{pmatrix} + t\begin{pmatrix}-2\\1\end{pmatrix}, \text{ where } t \in \mathbb{R}$$

Set these two equations equal to each other, i.e., $\overrightarrow{p_1q_2} = \overrightarrow{p_2q_1}$

$$\binom{1}{0} + s\binom{-1}{3} = \binom{2}{0} + t\binom{-2}{1} \implies \begin{matrix}1 - s = 2 - 2t\\ 3s = t\end{matrix} \implies 2(3s) - s = 1 \implies 5s = 1 \implies s = 1/5$$

The coordinates of r_{12} are given by

$$\overrightarrow{p_1q_2}\,(1/5) = \begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{5}\begin{pmatrix}-1\\3\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-1/5\\3/5\end{pmatrix} = \begin{pmatrix}4/5\\3/5\end{pmatrix}$$

Hence,

$$r_{12} = \boxed{(4/5, 3/5)}$$

For r_{13}: We need the intersection of p_1q_3 and p_3q_1:

$$\overrightarrow{p_1q_3}(s) := \begin{pmatrix}1\\0\end{pmatrix} + s\left[\begin{pmatrix}0\\5\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\right] = \begin{pmatrix}1\\0\end{pmatrix} + s\begin{pmatrix}-1\\5\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

$$\overrightarrow{p_3q_1}(t) := \begin{pmatrix}3\\0\end{pmatrix} + t\left[\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}3\\0\end{pmatrix}\right] = \begin{pmatrix}3\\0\end{pmatrix} + t\begin{pmatrix}-3\\1\end{pmatrix}, \text{ where } t \in \mathbb{R}$$

Set these two equations equal to each other, i.e., $\overrightarrow{p_1q_3} = \overrightarrow{p_3q_1}$

$$\binom{1}{0} + s\binom{-1}{5} = \binom{3}{0} + t\binom{-3}{1} \implies \begin{matrix}1 - s = 3 - 3t\\ 5s = t\end{matrix} \implies 1 - s = 3 - 3(5s) \implies 14s = 2 \implies s = 1/7$$

The coordinates of r_{13} are given by

$$\overrightarrow{p_1q_3}\,(1/7) = \begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{7}\begin{pmatrix}-1\\5\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-1/7\\5/7\end{pmatrix} = \begin{pmatrix}6/7\\5/7\end{pmatrix}$$

Hence,

$$r_{13} = \boxed{(6/7, 5/7)}$$

For r_{23}: We need the intersection of p_2q_3 and p_3q_2:

$$\overrightarrow{p_2q_3}(s) := \begin{pmatrix}2\\0\end{pmatrix} + s\left[\begin{pmatrix}0\\5\end{pmatrix} - \begin{pmatrix}2\\0\end{pmatrix}\right] = \begin{pmatrix}2\\0\end{pmatrix} + s\begin{pmatrix}-2\\5\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

$$\overrightarrow{p_3q_2}(t) := \begin{pmatrix}3\\0\end{pmatrix} + t\left[\begin{pmatrix}0\\3\end{pmatrix} - \begin{pmatrix}3\\0\end{pmatrix}\right] = \begin{pmatrix}3\\0\end{pmatrix} + t\begin{pmatrix}-3\\3\end{pmatrix}, \text{ where } t \in \mathbb{R}$$

Set these two equations equal to each other, i.e., $\overrightarrow{p_2q_3} = \overrightarrow{p_3q_2}$

$$\binom{2}{0} + s\binom{-2}{5} = \binom{3}{0} + t\binom{-3}{3} \implies \begin{matrix}2 - 2s = 3 - 3t\\ 5s = 3t\end{matrix} \implies 2 - 2s = 3 - 5s \implies 3s = 1 \implies s = 1/3$$

The coordinates of r_{23} are given by

$$\overrightarrow{p_2q_3}\,(1/3) = \begin{pmatrix}2\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}-2\\5\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}-2/3\\5/3\end{pmatrix} = \begin{pmatrix}4/3\\5/3\end{pmatrix}$$

Hence,

$$r_{13} = \boxed{(4/3, 5/3)}$$

Now, the equation for the line through r_{12} and r_{13} is given by:

$$\overrightarrow{r_{12}r_{13}}(s) := \begin{pmatrix}4/5\\3/5\end{pmatrix} + s\left[\begin{pmatrix}6/7\\5/7\end{pmatrix} - \begin{pmatrix}4/5\\3/5\end{pmatrix}\right] = \begin{pmatrix}4/5\\3/5\end{pmatrix} + s\begin{pmatrix}2/35\\4/35\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

Solving this vector equation in terms of x and y we get:

$$\left(\begin{smallmatrix}4/5\\3/5\end{smallmatrix}\right) + s\left(\begin{smallmatrix}2/35\\4/35\end{smallmatrix}\right) = \left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \implies \begin{matrix}4/5 + s2/35 = x\\3/5 + s4/35 = y\end{matrix} \implies 2x = 8/5 + 4/35s \implies y = 3/5 + 2x - 8/5 \implies \boxed{y = 2x - 1}$$

Finally, note that point r_{23} satisfies this equation since $\frac{5}{3} = 2\left(\frac{4}{3}\right) - 1$. Therefore, all three points r_{12}, r_{13} and r_{23} lie in the line $y = 2x - 1$, meaning that they are collinear.

(Ex. 4) Consider the three points $p_1 = (3,1), p_2 = (5,3), p_3 = (2,5)$, and the three points $q_1 = (-5,5), q_2 = (-3,1), q_3 = (-1,-4)$.

1. Show that the three lines p_1q_1, p_2q_2 and p_3q_3 are concurrent, and determine their common intersection.
2. Compute the intersection r_{ij} of the lines p_ip_j and q_iq_j for $i \neq j$
3. Show that the three points r_{12}, r_{13}, r_{23} are collinear.

Solution:

1. Let us first find the point of intersection between p_1q_1 and p_2q_2:

$$\overrightarrow{p_1q_1}(s) := \begin{pmatrix}3\\1\end{pmatrix} + s\left[\begin{pmatrix}-5\\5\end{pmatrix} - \begin{pmatrix}3\\1\end{pmatrix}\right] = \begin{pmatrix}3\\1\end{pmatrix} + s\begin{pmatrix}-8\\4\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

$$\overrightarrow{p_2q_2}(t) := \begin{pmatrix}5\\3\end{pmatrix} + t\left[\begin{pmatrix}-3\\1\end{pmatrix} - \begin{pmatrix}5\\3\end{pmatrix}\right] = \begin{pmatrix}5\\3\end{pmatrix} + t\begin{pmatrix}-8\\-2\end{pmatrix}, \text{ where } t \in \mathbb{R}$$

To find the intersection set these equal, i.e., $\overrightarrow{p_1q_1} = \overrightarrow{p_2q_2}$, which means:

$$\begin{pmatrix}3\\1\end{pmatrix} + s\begin{pmatrix}-8\\4\end{pmatrix} = \begin{pmatrix}5\\3\end{pmatrix} + t\begin{pmatrix}-8\\-2\end{pmatrix}, \quad \text{from which it follows:}$$

$$\begin{matrix}3 - 8s = 5 - 8t & & & & \implies 3 - [4 - 4t] = 5 - 8t \implies 12t = 6\\1 + 4s = 3 - 2t & \implies & 4s = 2 - 2t & \implies & 8s = 4 - 4t\end{matrix}$$

Therefore, $t = 1/2$ is the parameter for line $\overrightarrow{p_2q_2}$ for which it intersects with $\overrightarrow{p_1q_1}$. Coordinates of the intersection:

$$\overrightarrow{p_1q_1}\,(1/2) = \begin{pmatrix}5\\3\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-8\\-2\end{pmatrix} = \begin{pmatrix}5\\3\end{pmatrix} + \begin{pmatrix}-4\\-1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

So line $\overrightarrow{p_1q_1}$ intersects line $\overrightarrow{p_2q_2}$ at the point $\boxed{(1,2)}$. All that remains is to check whether this point is on the line p_3q_3, and indeed this is the case since:

$$\overrightarrow{p_3q_3}(r) := \begin{pmatrix}2\\5\end{pmatrix} + r\left[\begin{pmatrix}-1\\-4\end{pmatrix} - \begin{pmatrix}2\\5\end{pmatrix}\right] = \begin{pmatrix}2\\5\end{pmatrix} + r\begin{pmatrix}-3\\-9\end{pmatrix}, \text{ where } r \in \mathbb{R}$$

$$\overrightarrow{p_3q_3}(1/3) = \begin{pmatrix}2\\5\end{pmatrix} + \frac{1}{3}\begin{pmatrix}-3\\-9\end{pmatrix} = \begin{pmatrix}2\\5\end{pmatrix} + \begin{pmatrix}-1\\-3\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

So, the common intersection of p_1q_1, p_2q_2 and p_3q_3 is $\boxed{(1,2)}$.

2. Let us begin:

For r_{12}: We need the intersection of p_1p_2 and q_1q_2:

$$\overrightarrow{p_1p_2}(s) := \begin{pmatrix}3\\1\end{pmatrix} + s\left[\begin{pmatrix}5\\3\end{pmatrix} - \begin{pmatrix}3\\1\end{pmatrix}\right] = \begin{pmatrix}3\\1\end{pmatrix} + s\begin{pmatrix}2\\2\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

$$\overrightarrow{q_1q_2}(t) := \begin{pmatrix}-5\\5\end{pmatrix} + t\left[\begin{pmatrix}-3\\1\end{pmatrix} - \begin{pmatrix}-5\\5\end{pmatrix}\right] = \begin{pmatrix}-5\\5\end{pmatrix} + t\begin{pmatrix}2\\-4\end{pmatrix}, \text{ where } t \in \mathbb{R}$$

Set these two equations equal to each other, i.e., $\overrightarrow{p_1p_2} = \overrightarrow{q_1q_2}$

$$\binom{3}{1} + s\binom{2}{2} = \binom{-5}{5} + t\binom{2}{-4} \implies \begin{array}{l} 3+2s = -5+2t \\ 1+2s = 5-4t \end{array} \implies \begin{array}{l} 2s = -8+2t \\ 1-8+2t = 5-4t \end{array} \implies 6t = 12 \implies t = 2$$

The coordinates of r_{12} are given by

$$\overrightarrow{q_1q_2}\,(2) = \begin{pmatrix}-5\\5\end{pmatrix} + 2\begin{pmatrix}2\\-4\end{pmatrix} = \begin{pmatrix}-5\\5\end{pmatrix} + \begin{pmatrix}4\\-8\end{pmatrix} = \begin{pmatrix}-1\\-3\end{pmatrix}$$

Hence,

$$r_{12} = \boxed{(-1,-3)}$$

<u>For r_{13}:</u> We need the intersection of p_1p_3 and q_1q_3:

$$\overrightarrow{p_1p_3}(s) := \begin{pmatrix}3\\1\end{pmatrix} + s\left[\begin{pmatrix}2\\5\end{pmatrix} - \begin{pmatrix}3\\1\end{pmatrix}\right] = \begin{pmatrix}3\\1\end{pmatrix} + s\begin{pmatrix}-1\\4\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

$$\overrightarrow{q_1q_3}(t) := \begin{pmatrix}-5\\5\end{pmatrix} + t\left[\begin{pmatrix}-1\\-4\end{pmatrix} - \begin{pmatrix}-5\\5\end{pmatrix}\right] = \begin{pmatrix}-5\\5\end{pmatrix} + t\begin{pmatrix}4\\-9\end{pmatrix}, \text{ where } t \in \mathbb{R}$$

Set these two equations equal to each other, i.e., $\overrightarrow{p_1p_3} = \overrightarrow{q_1q_3}$

$$\binom{3}{1} + s\binom{-1}{4} = \binom{-5}{5} + t\binom{4}{-9} \implies \begin{array}{l} 3-s = -5+4t \\ 1+4s = 5-9t \end{array} \implies \begin{array}{l} s = 8-4t \\ 4(8-4t) = 4-9t \end{array} \implies 28 = 7t \implies t = 4$$

The coordinates of r_{13} are given by

$$\overrightarrow{q_1q_3}\,(4) = \begin{pmatrix}-5\\5\end{pmatrix} + 4\begin{pmatrix}4\\-9\end{pmatrix} = \begin{pmatrix}-5\\5\end{pmatrix} + \begin{pmatrix}16\\-36\end{pmatrix} = \begin{pmatrix}11\\-31\end{pmatrix}$$

Hence,

$$r_{13} = \boxed{(11,-31)}$$

<u>For r_{23}:</u> We need the intersection of p_2p_3 and q_2q_3:

$$\overrightarrow{p_2p_3}(s) := \begin{pmatrix}5\\3\end{pmatrix} + s\left[\begin{pmatrix}2\\5\end{pmatrix} - \begin{pmatrix}5\\3\end{pmatrix}\right] = \begin{pmatrix}5\\3\end{pmatrix} + s\begin{pmatrix}-3\\2\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

$$\overrightarrow{q_2q_3}(t) := \begin{pmatrix}-3\\1\end{pmatrix} + t\left[\begin{pmatrix}-1\\-4\end{pmatrix} - \begin{pmatrix}-3\\1\end{pmatrix}\right] = \begin{pmatrix}-3\\1\end{pmatrix} + t\begin{pmatrix}2\\-5\end{pmatrix}, \text{ where } t \in \mathbb{R}$$

Set these two equations equal to each other, i.e., $\overrightarrow{p_2p_3} = \overrightarrow{q_2q_3}$

$$\binom{5}{3} + s\binom{-3}{2} = \binom{-3}{1} + t\binom{2}{-5} \implies \begin{array}{l} 5-3s = -3+2t \\ 3+2s = 1-5t \end{array} \implies \begin{array}{l} t = (8-3s)/2 \\ 2+2s = -5(8-3s)/2 \end{array} \implies 44 = 11s \implies s = 4$$

The coordinates of r_{23} are given by

$$\overrightarrow{p_2p_3}\,(4) = \begin{pmatrix}5\\3\end{pmatrix} + 4\begin{pmatrix}-3\\2\end{pmatrix} = \begin{pmatrix}5\\3\end{pmatrix} + \begin{pmatrix}-12\\8\end{pmatrix} = \begin{pmatrix}-7\\11\end{pmatrix}$$

Hence,

$$r_{13} = \boxed{(-7,11)}$$

3. The equation for the line through r_{12} and r_{13} is given by:

$$\overrightarrow{r_{12}r_{13}}(s) := \begin{pmatrix}-1\\-3\end{pmatrix} + s\left[\begin{pmatrix}11\\-31\end{pmatrix} - \begin{pmatrix}-1\\-3\end{pmatrix}\right] = \begin{pmatrix}-1\\-3\end{pmatrix} + s\begin{pmatrix}12\\-28\end{pmatrix}, \text{ where } s \in \mathbb{R}$$

Solving this vector equation in terms of x and y we get:

$$\binom{-1}{-3} + s\binom{12}{-28} = \binom{x}{y} \implies \begin{array}{l} -1+12s = x \\ -3-28s = y \end{array} \implies s = (x+1)/12 \implies -3-28[(x+1)/12] = y \implies$$

Hence, $\boxed{y = \dfrac{-16}{3} - \dfrac{7}{3}x}$. Finally, note that point r_{23} satisfies this equation since $\dfrac{-16-7(-7)}{3} = \dfrac{-16+49}{3} = \dfrac{33}{3} = 11$. Therefore, all three points r_{12}, r_{13} and r_{23} lie in the line $y = \frac{-16}{3} - \frac{7}{3}x$, meaning that they are <u>collinear</u>.

(Ex. 5) Show that the midpoints of the edges of a quadrilateral in the plane are the vertices of a parallelogram. Is this still true if the quadrilateral lies in space?

Solution: Consider the quadrilateral formed by the four arbitrary points P_1, P_2, P_3 and P_4, oriented counterclockwise. Let these points have coordinates:

$$P_1 = \begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix}, P_2 = \begin{pmatrix} p_{21} \\ p_{22} \end{pmatrix}, P_3 = \begin{pmatrix} p_{31} \\ p_{32} \end{pmatrix}, \text{ and } P_4 = \begin{pmatrix} p_{41} \\ p_{42} \end{pmatrix}$$

Now, the edges of the quadrilateral are given by lines P_1P_4, P_1P_2, P_2P_3 and P_3P_4. The midpoints of these edges, which I will denote P_{mij}, have coordinates:

$$P_{m14} = \frac{1}{2}\left[\begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} + \begin{pmatrix} p_{41} \\ p_{42} \end{pmatrix}\right] \qquad P_{m12} = \frac{1}{2}\left[\begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} + \begin{pmatrix} p_{21} \\ p_{22} \end{pmatrix}\right]$$

$$P_{m23} = \frac{1}{2}\left[\begin{pmatrix} p_{21} \\ p_{22} \end{pmatrix} + \begin{pmatrix} p_{31} \\ p_{32} \end{pmatrix}\right] \qquad P_{m34} = \frac{1}{2}\left[\begin{pmatrix} p_{31} \\ p_{32} \end{pmatrix} + \begin{pmatrix} p_{41} \\ p_{42} \end{pmatrix}\right]$$

To show that these points form a parallelogram, it suffices to show that the lines $P_{m12}P_{m14}$ and $P_{m23}P_{m34}$ are parallel; and that the lines $P_{m14}P_{m34}$ and $P_{m12}P_{m23}$ are parallel. For that end, we can check that the director vectors (which I will denote $\overrightarrow{P_{mij}P_{mlk}}$) are multiple of each other. So, let us check:

$$\overrightarrow{P_{m12}P_{m14}} := \frac{1}{2}\left[\begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} + \begin{pmatrix} p_{41} \\ p_{42} \end{pmatrix}\right] - \frac{1}{2}\left[\begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} + \begin{pmatrix} p_{21} \\ p_{22} \end{pmatrix}\right] = \frac{1}{2}\left[\begin{pmatrix} p_{41} \\ p_{42} \end{pmatrix} - \begin{pmatrix} p_{21} \\ p_{22} \end{pmatrix}\right]$$

$$\overrightarrow{P_{m23}P_{m34}} := \frac{1}{2}\left[\begin{pmatrix} p_{31} \\ p_{32} \end{pmatrix} + \begin{pmatrix} p_{41} \\ p_{42} \end{pmatrix}\right] - \frac{1}{2}\left[\begin{pmatrix} p_{21} \\ p_{22} \end{pmatrix} + \begin{pmatrix} p_{31} \\ p_{32} \end{pmatrix}\right] = \frac{1}{2}\left[\begin{pmatrix} p_{41} \\ p_{42} \end{pmatrix} - \begin{pmatrix} p_{21} \\ p_{22} \end{pmatrix}\right]$$

This shows that lines $P_{m12}P_{m14}$ and $P_{m23}P_{m34}$ are parallel, since $\overrightarrow{P_{m12}P_{m14}} = 1 \cdot \overrightarrow{P_{m23}P_{m34}}$. For the other case:

$$\overrightarrow{P_{m14}P_{m34}} := \frac{1}{2}\left[\begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} + \begin{pmatrix} p_{41} \\ p_{42} \end{pmatrix}\right] - \frac{1}{2}\left[\begin{pmatrix} p_{31} \\ p_{32} \end{pmatrix} + \begin{pmatrix} p_{41} \\ p_{42} \end{pmatrix}\right] = \frac{1}{2}\left[\begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} - \begin{pmatrix} p_{31} \\ p_{32} \end{pmatrix}\right]$$

$$\overrightarrow{P_{m12}P_{m23}} := \frac{1}{2}\left[\begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} + \begin{pmatrix} p_{21} \\ p_{22} \end{pmatrix}\right] - \frac{1}{2}\left[\begin{pmatrix} p_{21} \\ p_{22} \end{pmatrix} + \begin{pmatrix} p_{31} \\ p_{32} \end{pmatrix}\right] = \frac{1}{2}\left[\begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} - \begin{pmatrix} p_{31} \\ p_{32} \end{pmatrix}\right]$$

This shows that lines $P_{m14}P_{m34}$ and $P_{m12}P_{m23}$ are parallel, since $\overrightarrow{P_{m14}P_{m34}} = 1 \cdot \overrightarrow{P_{m14}P_{m34}}$

Note that this result will hold if the quadrilateral lies in space (an even higher dimensions). This can be easily check by adding the appropriate components to the previous vectors and running through the same computations (which I won't show here to save space, but this is really very similar to the previous computations).

The following graphic shows an schematic of this proof in the planar case.