M436 - Introduction to Geometries - Homework 3

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(Ex. 1) Show that for any rational number $q \in \mathbb{Q}$, there are two distinct points P_1 and P_2 with integer coordinates such that the line through P_1 and P_2 intersects the x-axis in $(q, 0)$.

Solution: Let $q = \dfrac{a}{b} \in \mathbb{Q}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$. Choose $P_1 = (1, b-a)$ and $P_2 = (0, -a)$. Clearly, since $a, b \in \mathbb{Z}$, both P_1 and P_2 have integer coordinates. Let $\vec{l}_1$ be the line through points P_1 and P_2, and let $\vec{l}_2$ be a parametrization of the x-axis. Then:

$$\vec{l}_1(s) := \begin{pmatrix} 1 \\ b-a \end{pmatrix} + s\left[\begin{pmatrix} 0 \\ -a \end{pmatrix} - \begin{pmatrix} 1 \\ b-a \end{pmatrix}\right] = \begin{pmatrix} 1 \\ b-a \end{pmatrix} + s\begin{pmatrix} -1 \\ -b \end{pmatrix}, \text{ where } s \in \mathbb{R}$$

$$\vec{l}_2(t) := t\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ where } t \in \mathbb{R}$$

To find the intersection set these equal, i.e., $\vec{l}_1 = \vec{l}_2$, which means:

$$\begin{pmatrix} 1 \\ b-a \end{pmatrix} + s\begin{pmatrix} -1 \\ -b \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{from which it follows:}$$

$$\begin{matrix} 1 - s = t \\ b - a - sb = 0 \end{matrix} \implies \begin{matrix} \\ s = 1 - a/b \end{matrix} \implies 1 - (1 - a/b) = t \implies t = a/b$$

Therefore, $t = \dfrac{a}{b}$ is the parameter for line $\vec{l}_2$ for which it intersects with $\vec{l}_1$. The coordinates of the intersection are:

$$\vec{l}_2\,(a/b) = \frac{a}{b}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a/b \\ 0 \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix}$$

So line $\vec{l}_1$ intersects line $\vec{l}_2$ at the point $\boxed{(q, 0)}$, showing the result.

(Ex. 2) Show that the set of pairs $\{(a, b) : a, b \in \mathbb{F}_3\}$ becomes a field by defining

$$(a, b) + (a', b') = (a + a', b + b')$$

$$(a, b) \cdot (a', b') = (aa' - bb', ab' + a'b)$$

If we write $1 = (1, 0)$ and $i = (0, 1)$, we can also write $a + bi = (a, b)$, and have the familiar identity $i^2 = -1$.
Hint for the multiplicative inverse:

$$\frac{1}{(a, b)} = \frac{(a, -b)}{a^2 + b^2}$$

Why do we not divide by 0? Does this also work if we replace $\mathbb{F}_3$ with $\mathbb{F}_5$?

Solution: To check that the set of pairs, call it $S = \{(a, b) : a, b \in \mathbb{F}_3\}$ is a field, we would have to check:

1) $(S, +, 0)$ is an abelian group, where $0 = (0, 0)$.
2) $(S^*, \cdot, 1)$ is an abelian group, where $1 = (1, 0)$.
3) For every $s \in S$, we must have $0 \cdot s = s \cdot 0 = 0$.
4) For every $s_1, s_2, s_3 \in S$, we must have $s_1 \cdot (s_2 + s_3) = s_1 \cdot s_2 + s_1 \cdot s_3$. Commutativity takes care of the other way.

Let us check:

1) To check that $(S, +, 0)$ is an abelian group, we need to verify the following:
   - (1.1) S is <u>closed</u> under $+$, since $\mathbb{F}_3$ is closed under addition modulo 3.
   - (1.2) $+$ is <u>associative</u> since the underlying operation is associative in $\mathbb{F}_3$.
   - (1.3) $+$ is <u>commutative</u> since the underlying operation is commutative in $\mathbb{F}_3$.
   - (1.4) $(0, 0)$ is the additive <u>identity</u> since $(a, b) + (0, 0) = (a + 0, b + 0) = (a, b)$.

(1.5) Let $(a,b) \in S$. Then its additive <u>inverse</u> is $(-a,-b)$ since $(a,b)+(-a,-b) = (a+(-a), b+(-b)) = (0,0)$

2) To check that $(S^*, \cdot, 1)$ is an abelian group, we need to verify the following:

(1.1) S^* is <u>closed</u> under $\cdot$, since $\mathbb{F}_3$ is closed under addition and multiplication modulo 3.

(1.2) $+$ is <u>associative</u> since the underlying operations (addition and multiplication) are associative in $\mathbb{F}_3$.

(1.3) $+$ is <u>commutative</u>: Let $(a,b), (a',b') \in S$. Then,

$$(a,b)\cdot(a',b') = (aa'-bb', ab'+a'b) = (a'a-b'b, a'b+ab') = (a',b')\cdot(a,b)$$

(1.4) $(1,0)$ is the multiplicative <u>identity</u> since $(a,b)\cdot(1,0) = (a1-b0, a0+1b) = (a-0, 0+b) = (a,b)$.

(1.5) Let $(a,b) \in S^*$, i.e., $(a,b) \neq (0,0)$. Then its <u>inverse</u> is $\dfrac{1}{(a,b)} = \dfrac{(a,-b)}{a^2+b^2}$ since

$$\begin{aligned}(a,b)\cdot\frac{1}{(a,b)} = (a,b)\cdot\frac{(a,-b)}{a^2+b^2} &= \left(a\frac{a}{a^2+b^2} - b\frac{-b}{a^2+b^2}, a\frac{-b}{a^2+b^2} + \frac{a}{a^2+b^2}b\right)\\ &= \left(\frac{a^2+b^2}{a^2+b^2}, \frac{-ab}{a^2+b^2} + \frac{ab}{a^2+b^2}\right)\\ &= (1,0)\end{aligned}$$

Note that we do not divide by 0 here because $a^2+b^2 \neq 0$ whenever $a,b \in \mathbb{F}_3^*$. Indeed,

$$1^2+0^2 = 1, 1^2+1^2 = 2, 1^2+2^2 = 2, 2^2+0^2 = 1, 2^2+2^2 = 2 \text{ (commutativity takes care of the rest)}$$

3) Let $(a,b) \in S$. Then, $(a,b)\cdot(0,0) = (a0-b0, a0+0b) = (0,0)$

4) Let $(a,b), (c,d), (e,f) \in S$. Then:

$$(a,b)\cdot[(c,d)+(e,f)] = (a,b)\cdot(c+e, d+f) = (a(c+e)-b(d+f), a(d+f)+(c+e)b)$$

$$\begin{aligned}[(a,b)\cdot(c,d)]+[(a,b)\cdot(e,f)] = (ac-bd, ad+cb)+(ae-bf, af+eb) &= (ac-bd+ae-bf, ad+cb+af+eb)\\ &= (a(c+e)-b(d+f), a(d+f), (c+e)b)\end{aligned}$$

Hence, the distributive property holds.

Note that this <u>does not</u> work if we replace $\mathbb{F}_3$ with $\mathbb{F}_5$, because we would divide by zero for some multiplicative inverses, for example $1^2+2^2 = 5 = 0$, and in particular the pair $(1,2)$ would not have a multiplicative inverse.

(Ex. 3) In this exercise, we will study the affine plane $\mathbb{F}_3^2$.

1. How many points are in $\mathbb{F}_3^2$?
2. How many lines are in $\mathbb{F}_3^2$?
3. How many lines are in $\mathbb{F}_3^2$ that pass through the origin $(0,0)$?
4. How many points lie on each line?

Solution: Let $\mathbb{F}_3^2 = \{\binom{a}{b} : a,b \in \mathbb{F}_3\}$. Then

1. There are 3 choices for the first coordinate a and 3 choices for the second coordinate b. By the multiplication rule, we have $\boxed{|\mathbb{F}_3^2| = 3\times 3 = 9}$, i.e., there are 9 points in $\mathbb{F}_3^2$.
2. Through every 2 distinct points there is exactly one line. Hence, there are $\boxed{\binom{9}{2} = \frac{9\cdot 8}{2} = 36}$ lines.
3. If we fix the point $(0,0)$, then we can choose 4 other non-parallel vectors as direction vectors to make a line through this point, i.e., there are $\boxed{\binom{4}{1} = 4}$ lines through $(0,0)$. This follows from the fact that if two lines have parallel direction vector and have a common point then they are the same line.
4. Three points lie on each line. By definition, the line through the point p in the direction v is the set $\{p+tv : t \in \mathbb{F}_3\}$. If $t = 0$, then we are in p. The only other two possible values for t are 1 and 2, so that we get three distinct points p, $p+v$ and $p+2v$, provided that $v \neq 0$.

(Ex. 4) In this exercise, we will study the special linear group $SL_2(\mathbb{F}_3)$.

1. How many 2×2 matrices with entries in $\mathbb{F}_3$ have rank 0?
2. How many 2×2 matrices with entries in $\mathbb{F}_3$ have rank 1?
3. How many 2×2 matrices with entries in $\mathbb{F}_3$ have rank 2?
4. How many elements are in $SL_2(\mathbb{F}_3)$?

Solution:

1. There is only $\boxed{1}$ matrix of rank 0, i.e., the null matrix.
2. A rank 1 matrix means that there is only one linearly independent vector (viewed as a column vector), which will mean that the other vector is a multiple of this vector. To construct a two dimensional vector we have $3 \times 3 = 9$ choices, however we must exclude the vector $(0, 0)$ or otherwise we would get the null matrix. Therefore, there are 8 non-zero vectors in $\mathbb{F}_3^2$. Each of these vectors has three different multiples given by multiplication by $0, 1$ or 2. So there are $8 \times 3 = 24$ matrices of rank 1 without the $(0, 0)$ vector. Using the $(0, 0)$ vector we can get 8 more matrices of rank 1. Therefore, there are $\boxed{24 + 8 = 32}$
3. A rank 2 matrix means that there are two linearly independent vectors in the matrix. From part 2. we know that there are 8 non-zero vectors. To construct a rank 2 matrix we must choose two different vectors. Given a vector we must not choose any of its 3 multiples. Therefore, there are $\boxed{8 \times (9 - 3) = 8 \times 6 = 48}$ matrices of rank 2. Note that there are $3^4 = 81$, 2×2 matrices in total and we have accounted for all of them since $1 + 32 + 48 = 81$.
4. By definition $SL_2(\mathbb{F}_3) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{F}_3, det(A) = 1 \right\}$. To count how many matrices are in this set, let us partition the space into three partitions according to the value for the first entry a.

$a = 0$, in this case the determinant condition becomes $0d - bc = 1 \iff -bc = 1 \iff b = -1/c$. There are three choices for the value of d but only two for the value of c since we cannot divide by zero. Once c is chosen the value of b is determined by the previous equation. Hence, there are $3 \times 2 = 6$ matrices in $SL_2(\mathbb{F}_3)$ with the first entry being 0.

$a = 1$, in this case the determinant condition becomes $1d - bc = 1 \iff d = 1 + bc$. There are three choices for the value of b and three choices for the value of c. Once these choices are made the value of d is determined by the previous equation. Hence, there are $3 \times 3 = 9$ matrices in $SL_2(\mathbb{F}_3)$ with the first entry being 1.

$a = 2$, in this case the determinant condition becomes $2d - bc = 1 \iff d = (1 + bc)2^{-1}$. Again, There are three choices for the value of b and three choices for the value of c. Once these choices are made the value of d is determined by the previous equation. Hence, there are $3 \times 3 = 9$ matrices in $SL_2(\mathbb{F}_3)$ with the first entry 2.

Since the three previous cases partition the space, their sum is the number of elements in $SL_2(\mathbb{F}_3)$, i.e.,

$$\boxed{|SL_2(\mathbb{F}_3)| = 6 + 9 + 9 = 24}$$

(Ex. 5) Show that in any affine plane $\mathbb{F}^2$ over a field $\mathbb{F}$, two lines are either equal, intersect in one point, or are disjoint and parallel. Instructions: any line is given as a set $\{p + tv : t \in \mathbb{F}\}$ where $p \in \mathbb{F}^2$ is a point and $v \in \mathbb{F}^2$ is a non-zero direction vector. Two lines are parallel if they are given by proportional direction vectors. Show

1. If two lines are parallel and have at least one point in common, they are equal. It suffices that one line is contained in other.
2. If two lines are non-parallel, they meet in precisely one point. Use that two independent vectors in $\mathbb{F}^2$ span $\mathbb{F}^2$.

Proof: Let two lines be given: $\{p_1 + t_1 v_1\}$ and $\{p_2 + t_2 v_2\}$. Let us work each case.

1. Suppose the two lines are parallel. Then, $v_2 = sv_1$, for some $s \in \mathbb{F}$. We can replace this in the equation: $p_1 + t_1 v_1 = p_2 + t_2 v_2 \iff t_1 v_1 - t_2 v_2 = p_2 - p_1 \iff t_1 v_1 - t_2 s v_1 = p_2 - p_1 \iff (t_1 - t_2 s) v_1 = p_2 - p_1$. There are two cases: (1) $p_2 - p_1$ is a linearly dependent of v_1, in which case $p_2 - p_1 = av_1$ for some $a \in \mathbb{F}$. Replace: $(t_1 - t_2 s) v_1 = av_1 \implies a = (t_1 - t_2 s)$. Solving for t_2 shows that the lines are identical. (2) $p_2 - p_1$ is linearly independent of v_1. In this case the equation $(t_1 - t_2 s) v_1 = p_2 - p_1 \implies (t_1 - t_2 s) v_1 - (p_2 - p_1) = 0$, which would imply that $-1 = 0$, a contradiction, and hence there are no common points to these two lines, i.e., the lines are parallel.
2. Suppose that the lines are non-parallel. Then the direction vectors v_1, v_2 are linearly independent. Two independent vectors in a 2-dimensional space span the whole space. In particular, $\{v_1, v_2\}$, form a basis for $\mathbb{F}^2$. In this case, we can always solve for t_1 and t_2 in the equation: $p_1 + t_1 v_1 = p_2 + t_2 v_2 \iff t_1 v_1 - t_2 v_2 = p_2 - p_1$. Moreover, this solution is unique, giving us the common intersection point of the two lines.