— M436 — Homework Assignment 3 —

Due: Friday, September 19, in class.

Each problem is worth 20 points. Please show all your work.

Exercise 1

Show that for any rational number $q \in \mathbf{Q}$, there are two distinct points P_1 and P_2 with integer coordinates such that the line through P_1 and P_2 intersects the x-axis in $(q, 0)$.

Exercise 2

Show that the set of pairs $\{(a, b) : a, b \in \mathbf{F}_3\}$ becomes a field by defining $(a, b) + (a'b') = (a + a', b + b')$ and $(a, b) \cdot (a'b') = (aa' - bb', ab' + a'b)$. If we write $1 = (1, 0)$ and $i = (0, 1)$, we can also write $a + bi = (a, b)$, and have the familiar identity $i^2 = -1$. Hint for the multiplicative inverse:

$$\frac{1}{(a, b)} = \frac{(a, -b)}{a^2 + b^2}$$

Why do we do not divide by 0? Does this also work if we replace $\mathbf{F}_3$ with $\mathbf{F}_5$?

Exercise 3

In this exercise, we will study the affine plane $\mathbf{F}_3^2$.

1. How many point are in $\mathbf{F}_3^2$?
2. How many lines are in $\mathbf{F}_3^2$?
3. How many lines are in $\mathbf{F}_3^2$ that pass through the origin $(0, 0)$?
4. How many points lie on each line?

Exercise 4

In this exercise, we will study the special linear group $SL_2(\mathbf{F}_3)$.

1. How many 2×2 matrices with entries in $\mathbf{F}_3$ have rank 0?
2. How many 2×2 matrices with entries in $\mathbf{F}_3$ have rank 1?
3. How many 2×2 matrices with entries in $\mathbf{F}_3$ have rank 2?
4. How many elements are in $SL_2(\mathbf{F}_3)$?

Exercise 5

Show that in any affine plane $\mathbf{F}^2$ over a field $\mathbf{F}$, two lines are either equal, intersect in one point, or are disjoint and parallel. Instructions: Any line is given as a set $\{p + tv : t \in \mathbf{F}\}$ where $p \in \mathbf{F}^2$ is a point and $v \in \mathbf{F}^2$ is a non-zero direction vector. Two lines are parallel if they are given by proportional direction vectors. Show

1. If two lines are parallel and have at least one point in common, they are equal. It suffices that one line is contained in the other.
2. If two lines are non-parallel, they meet in precisely one point. Use that two independent vectors in $\mathbf{F}^2$ span $\mathbf{F}^2$.