M436 - Introduction to Geometries - Homework 8

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(Ex. 1) Find all quaternions $q = a + bi + cj + dk$ such that $q^2 = -13 + 6i - 2j + 4k$.

Solution: To find all such quaternion we solve the equation:

$$q^2 = (a + bi + cj + dk)(a + bi + cj + dk) = -13 + 6i - 2j + 4k \Longrightarrow$$

$$a^2+abi+acj+adk+abi+b^2i^2+bcij+bdik+acj+bcji+c^2j^2+cdjk+adk+bdki+cdkj+d^2k^2 = -13+6i-2j+4k \Longrightarrow$$

$$a^2 + 2abi + acj + adk - b^2 + bck - bdj + acj - bck - c^2 + cdi + adk + bdj - cdi - d^2 = -13 + 6i - 2j + 4k \Longrightarrow$$

$$a^2 - b^2 - c^2 - d^2 + 2abi + 2acj + 2adk = -13 + 6i - 2j + 4k \Longrightarrow$$

$$\left\{\begin{array}{lclcl} a^2 - b^2 - c^2 - d^2 = -13 & \Longrightarrow & a^2 - b^2 - c^2 - d^2 = -13 & \Longrightarrow & a^2 - b^2 - c^2 - d^2 = -13 \\ 2ab = 6 & & ab = 3 & & b = 3/a \\ 2ac = -2 & & ac = -1 & & c = -1/a \\ 2ad = 4 & & ad = 2 & & d = 2/a \end{array}\right\}$$

Replacing the last three equations into the first one we obtain: $a^2 - \frac{9}{a^2} - \frac{1}{a^2} - \frac{4}{a^2} = -13 \iff a^2 - \frac{14}{a^2} = -13$, and hence (note that $a \neq 0, b \neq 0, c \neq 0, d \neq 0$)

$$a^4 + 13a^2 - 14 = 0$$

Note that both $a = 1$ and $a = -1$ are roots of the polynomial $a^4 + 13a^2 - 14$, since by inspection $1^4 + 13(1)^2 - 14 = 0$. So we divide the polynomial by $(a - 1)$ to deduce that $a^4 + 13a^2 - 14 = (a^3 + a^2 + 14a + 14)(a - 1)$. We can further divide the polynomial $a^3 + a^2 + 14a + 14$ by $a + 1$ to get that $a^3 + a^2 + 14a + 14 = (a^2 + 14)(a + 1)$. Combining these results we get:

$$a^4 + 13a^2 - 14 = (a^2 + 14)(a + 1)(a - 1) = (a + \sqrt{14}i)(a - \sqrt{14}i)(a + 1)(a - 1)$$

And so the possible values for a are $a = \pm 1, a = \pm\sqrt{14}i$. Each of these 4 values gives a value for b, c and d. The following table summarizes this information:

a	b	c	d
1	3	-1	2
-1	-3	1	-2
$-\sqrt{14}i$	$\frac{3i}{\sqrt{14}}$	$-\frac{i}{\sqrt{14}}$	$\sqrt{\frac{2}{7}}i$
$\sqrt{14}i$	$-\frac{3i}{\sqrt{14}}$	$\frac{i}{\sqrt{14}}$	$-\sqrt{\frac{2}{7}}i$

(Ex. 2) Given a spherical triangle $\triangle$ with angles $0 < \alpha, \beta, \gamma < \pi$. For which choice of angles is there a tiling of the sphere S^2 by triangles with the same angles as $\triangle$ such that neighboring triangles are symmetric with respect to their shared edge?

Solution: Let α, β, γ be the angles of a spherical triangle. Let us consider a tiling of the sphere by this triangle. If the triangle closes when tiling the sphere then $\alpha = \frac{2\pi}{n}$, where $n \geq 3$ and $n \in \mathbb{Z}$. Moreover, since we want the triangle to be such that neighboring triangles are symmetric with respect to their shared edge, the same reasoning applies to β and γ, and so let us write these conditions as:

$$\alpha = \frac{2\pi}{A}, \beta = \frac{2\pi}{B}, \gamma = \frac{2\pi}{C}, \quad \text{where } A, B, C \in \mathbb{Z} \text{ and } A, B, C \geq 3$$

An spherical triangle is such that $\alpha + \beta + \gamma > \pi$. This means that:

$$\alpha + \beta + \gamma > \pi \Longrightarrow \frac{2\pi}{A} + \frac{2\pi}{B} + \frac{2\pi}{C} > \pi \iff 2\pi\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) > \pi \iff \frac{1}{A} + \frac{1}{B} + \frac{1}{C} > \frac{1}{2}$$

To classify all spherical triangles, it suffices to find solutions to the last equation, i.e., find suitable integers A, B, C. We do this by cases:

Suppose <u>A is odd</u>. Then the triangle is isosceles which means that $B = C$.

Suppose A and B are odd. Then, by previous case it follows that $B = C$ and $A = C$ so that $A = B = C$. So,

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} > \frac{1}{2} \Longrightarrow \frac{3}{A} > \frac{1}{2} \iff A < 6$$

Since A is odd, we have solutions: $A = B = C = 3$ or $A = B = C = 5$. The first case implies that $\alpha = \beta = \gamma = 120°$ and the second case implies that $\alpha = \beta = \gamma = 72°$

Suppose A is odd and $B = C$ is even. Replacing A, B, C by $A, 2b, 2b$. You find that:

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} = \frac{1}{A} + \frac{1}{2b} + \frac{1}{2b} = \frac{1}{A} + \frac{2}{2b} = \frac{1}{A} + \frac{1}{b} > \frac{1}{2} \Longrightarrow \frac{1}{b} > \frac{1}{2} - \frac{1}{A}$$

If $A = 3$, then $b < 6$. Since $B = 2b \geq 3 \Longrightarrow b \geq 3/2 \Longrightarrow B \geq 2$. Hence, $b = 2, 3, 4$ or 5 so that $B = C = 4, 6, 8, 10$. This yields triangles $(3,4,4), (3,6,6), (3,8,8), (3,10,10)$ for corresponding angles $(\alpha, \beta, \gamma) = (120°, 90°, 90°), (120°, 60°, 60°), (120°, 45°, 45°), (120°, 36°, 36°)$

Suppose A, B, C are all even. Then, $A = 2a, B = 2b, C = 2c$. This implies that

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} > \frac{1}{2} \Longrightarrow \frac{1}{2a} + \frac{1}{2b} + \frac{1}{2c} > \frac{1}{2} \iff \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) > \frac{1}{2} \iff \frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$$

Note that if $a = b = c = 3$, then $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$, so this case does not yield a spherical triangle.

Hence, it must be that one of a, b and c is less than 3. Without loss of generality, assume that $a = 2$ and that $a \leq b \leq c$ Here we have various sub cases:

$b = 2$. In this case: $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \Longrightarrow \frac{1}{2} + \frac{1}{2} + \frac{1}{c} > 1 \Longrightarrow \frac{1}{c} > 0$, so c can be any number bigger than 3. In short, this case yields triangles of the form $(2, 2, c)$ with $c > 3$, i.e. triangles with angles: $\alpha = \beta = 90°$ and $\gamma > 60°$.

$b > 4$ is impossible and thus, $b = 3$. From this we deduce: $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2} + \frac{1}{3} + \frac{1}{c} > 1 \Longrightarrow c < 6$, so the only options are $c = 3, 4$ or 5. This case yields triangles $(2,3,3), (2,3,4), (2,3,5)$ corresponding to angles $(\alpha, \beta, \gamma) = (90°, 60°, 60°), (90°, 60°, 45°), (90°, 60°, 36°)$

(Ex. 3) In the previous exercise, you should have found an example with $\alpha = \beta = \gamma = 72°$. Consider one of the triangles, and denote the reflections in S^2 at the edges of this triangle by ρ, ϕ, ψ. Let G be the subgroup of $O(3)$ generated by these reflections. Draw the Cayley graph of G with respect to the set $\Gamma = \{\rho, \phi, \psi\}$.

Solution: By definition, the Cayley Graph of G w.r.t Γ has G as vertices and a is connected to b if and only if there is $g \in \Gamma$ s.t. $b = a \cdot g$. As mentioned in class, we will consider the case of the tetrahedron. In this case, the subgroup of $O(3)$ generated by these reflection correspond to all triangles tilting the sphere. There are 24 such triangles. Then the Cayley graph is given by:

(Ex. 4) Determine all linear automorphisms of $\mathbb{F}_2^n$ (i.e. invertible $n \times n-$matrices with entries in $\mathbb{F}_2$) that are isometries with respect to the Hamming distance.

Solution: First, note that for $n = 1$, an invertible matrix corresponds just to the number 1, which clearly preserves the Hamming distance.
For $n \geq 2$, all linear automorphisms of $\mathbb{F}_2^n$ that are isometries with respect to the Hamming distance (d_H) correspond exactly to permutation matrices. Let us show both directions of this argument, i.e., $(i)(\Longrightarrow)$ A permutation matrix in $\mathbb{F}_2^n$ preserves Hamming distance and $(ii)(\Longleftarrow)$ A Hamming preserving distance in $\mathbb{F}_2^n$ is a permutation matrix.

(i) Let A be a permutation matrix over $\mathbb{F}_2$. Clearly, A is an invertible $n \times n$ matrix since its determinant is just the determinant of the identity with a possible change of sign. Now, take two points p, q in $\mathbb{F}_2^n$. Suppose that $d_H(p, q) = m$. This means that there are m different entries between p and q.
We know that $Ap = p'$, where p' is just a permutation of the entries of p. Likewise $Aq = q'$ where q' is a permutation of q. The key observation is that both p' and q' are permuted in the same entries by A. Therefore, p' and q' differ in the same number of entries as p and q. In symbols, $d_H(p', q') = d_H(Ap, Aq) = m = d_H(p, q)$. Hence, a permutation matrix preserves the Hamming distance.

(ii) Here is easier to argue for the contrapositive, i.e., if an invertible $n \times n$ matrix with entries in $\mathbb{F}_2$ is not a permutation matrix, then it does not preserve the Hamming distance.
Let A be an invertible $n \times n$ matrix with entries in $\mathbb{F}_2$ that is not a permutation matrix. Then, there is at least one column of A, say column i, that has at least 2 ones. Let $\{e_i\}$ be the canonical basis for $\mathbb{F}_2$. Then $d_H(e_i, e_j) = 2$ for any $i \neq j$. Note that Ae_i is just selecting the ith column of A. Moreover, there must be another column of A, say column j that differs in at least 3 positions with column i or otherwise the matrix would not be invertible because said column could be written as the linear combination of two or more columns of A. Hence, $d_H(Ae_i, Ae_j) \geq 3 \neq d_H(e_i, e_j) = 2$, and so this matrix A does not preserve the Hamming distance.

(Ex. 5) Let q be an imaginary unit quaternion, i.e. $|q| = 1$ and $\bar{q} = -q$. Show that the unit quaternions p such that $pq\bar{p} = q$ have the form $p = r + sq$ with $r, s \in \mathbb{R}$ and $r^2 + s^2 = 1$.

Solution: Let the unit quaternion p have the form $p = r + sq$ with $r, s \in \mathbb{R}$ and $r^2 + s^2 = 1$; and q a unit quaternion. Then:

$$
\begin{aligned}
pq\bar{p} &= (r+sq)q\overline{(r+sq)} && \text{by hypothesis on } p \\
&= (r+sq)q(\bar{r}+\overline{sq}) && \text{by properties of the conjugate} \\
&= (r+sq)q(r+s\bar{q}) && \text{by properties of the conjugate or real numbers} \\
&= (r+sq)q(r+s(-q)) && \text{by hypothesis on } q \\
&= (r+sq)q(r-sq) && \text{rearranging terms} \\
&= (rq+sq^2)(r-sq) && \text{distributive property} \\
&= (rq-s)(r-sq) && \text{since } q \text{ is a quaternion it follows } q^2=-1 \\
&= (r^2q-rsq^2-sr+s^2q) && \text{distributing and noting that reals commute with quaternions} \\
&= (r^2+s^2)q+rs-sr && \text{since } q \text{ is a quaternion it follows } q^2=-1 \\
&= q && \text{since } rs=sr \text{ and by hypothesis } r^2+s^2=1
\end{aligned}
$$

Hence, if $p = r + sq$ then $\boxed{pq\bar{p} = q}$