— M436 — Homework Assignment 9 —

Due: Friday, November 14, in class. Each problem is worth 20 points. Please show all your work. Homework that is illegible or discourages the reader otherwise from looking at it will be returned ungraded.

All problems below will use the stereographic projection in the form.

$$\sigma : S^n \to \mathbf{R}^n$$

$$(p_1, ...p_{n+1}) \mapsto \frac{1}{1 - p_{n+1}}(p_1, ...p_n)$$

Note that there are other conventions.

Exercise 1

Define a map $\iota : \mathbf{C} \to \mathbf{C}$ by first using the inverse stereographic projection to obtain a point in S^2, then applying a reflection at the plane $z = 0$, and finally applying the stereographic projection to get a point in $\mathbf{R}^2 = \mathbf{C}$. Show that this map is given by $z \mapsto 1/\overline{z}$.

Exercise 2

Let $p \in S^n \subset \mathbf{R}^{n+1}$ be a unit vector, and d be a real number. Recall that the set $\{x : p \cdot x = d\}$ describes a hyperplane perpendicular to n at distance d from the origin. Show that the intersection of this hyperplane wit S^n is mapped by the stereographic projection to a sphere in $\mathbf{R}^{n-1}$ with center at $\frac{1}{d-p_n}(p_1, ..., p_{n-1})$ and radius $\frac{\sqrt{1-d^2}}{|p_{n+1}-d|}$. Hint: We did the special case $d = 0$ in class.

Exercise 3

The cube with vertices at $\frac{1}{\sqrt{3}}(\pm 1, \pm 1, \pm 1)$ leads to a circle packing of S^2 with 8 circles by taking as centers of the circles the vertices of a cube, and making them so large and of equal radius so that they touch. When you stereographically project them into the

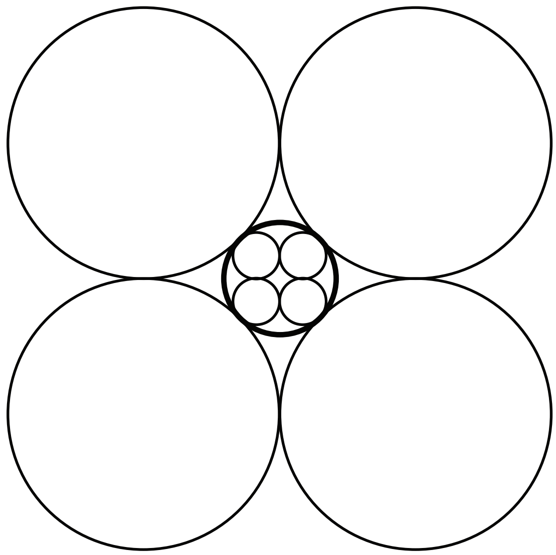

Figure 1

plane, you get a figure like the one above, where the fat circle in the center is the unit circle. Find the radii and centers of the other projected circles. Hint: First describe the circles as intersections of the sphere with suitable planes, then use the previous exercise.

Exercise 4

Show that two circles with centers at $p, q \in \mathbf{R}^2$ and radii $r, s > 0$ intersect at an angle ϕ with

$$\cos\phi = \frac{r^2 + s^2 - |p-q|^2}{2rs}$$

This angle of intersection is defined as the angle between the vectors $x - p$ and $x - q$ for a point x that lies on both circles. Hint: Expand $|(x-p) - (x-q)|^2$.

Exercise 5

Show that the dihedral angle of the regular Euclidean octahedron is approximately 109.47°.