— M436 — Final Exam —

The following five problems are each worth 40 points. Notes are encouraged. Please show all your work.

Problem 1

A spherical quadrilateral has edges that are pieces of great circles that are shorter than half a circumference. It tiles the sphere by reflection at its edges. Two of its angles are 72°. How many of these quadrilaterals are needed to tile the sphere?

Problem 2

Here are the rules for a lottery:

- You can purchase any number of tickets.
- Each ticket has the numbers 1–14 printed on it.
- To validate a ticket, you mark three different numbers from 1 to 14.
- After all tickets are validated, three different numbers from 1 to 14 are chosen randomly.
- Any ticket that has at least two numbers right will win a price in the lottery.

You will show that by purchasing 14 tickets and selecting the numbers wisely, you can ensure to have at least one winning ticket. Draw two Fano planes, and label the vertices in the first plane 1 through 7, and in the second plane 8 through 14. For each of the 14 lines in both Fano planes, purchase a ticket and mark on this ticket the numbers of the vertices that are on that line.

Explain why you must have at least one winning ticket.

Finite Math Note: The probability to win with a single ticket is about 9.3%. When you fill out 14 tickets randomly, the probability that at least one of your tickets wins becomes 74.67%. So what do we learn? Finite Geometry beats Finite Math!

Problem 3

The figure shows the projective plane over the finite field $\mathbf{F}_5$ with 5 elements, with the orderly arranged square depicting the affine plane $z = 1$. The six marked points are the points of the conic $\{(x : y : z) : x^2 + y^2 = z^2\}$, and the line is the tangent line of the conic at the point $(2 : 1 : 0)$. Note that parallel lines are in fact only a single line, passing through the same point at infinity.

Draw the tangent lines through the other five points of the conic. Explain in one case how you found that tangent line. You will observe that no three different tangent lines can meet in a single point. Show that this is the case for a general, non-degenerate conic in any projective plane over an arbitrary field $\mathbf{F}$. Hint: Look at the dual projective plane.

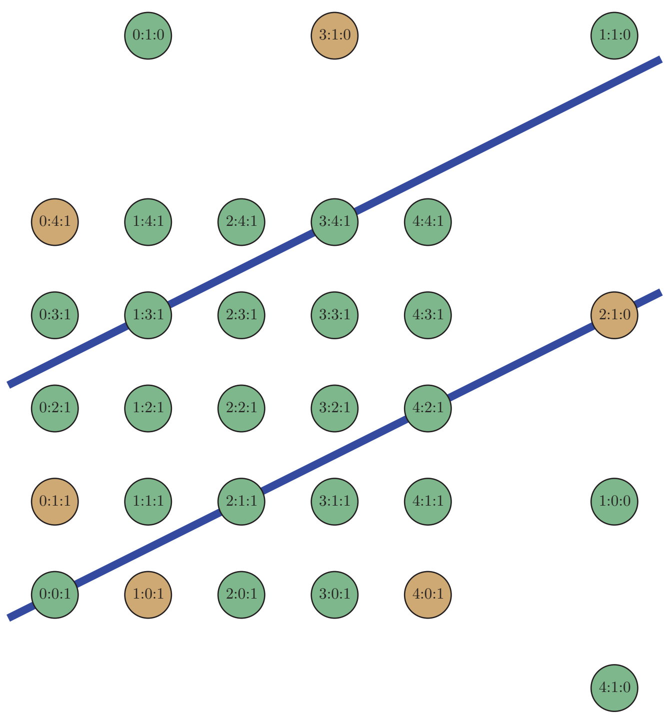

Figure 1 A conic in $\mathbf{F}_5\mathbf{P}^2$

Problem 4

Show that the Mobius transformation

$$f(z) = \frac{-5z+8}{z-7}$$

moves points that lie on the geodesic that foots on the real axis at -2 and 4 by the hyperbolic distance $\cosh^{-1}(5/3)$.

Problem 5

Let

$$SL_2(\mathbf{F}_3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbf{F}_3, ad - bc \equiv 1 \pmod 3 \right\}$$

be the set of 2×2 matrices with entries in the field of 3 elements and determinant 1 (modulo 3). For each such matrix, we associate the Möbius transformation

$$f(z) = \frac{az+b}{cz+d} \ .$$

You obtain 12 different Möbius transformation this way; they form a group under composition, using modular arithmetic for the basic operations involving the coefficients. They can be considered as maps of the set $\mathbf{F}_3 \cup \{\infty\}$ with the proviso that any attempted division by 0 has the result ∞, and $f(\infty) = a/c$. Draw the Cayley graph of this group with respect to the set

$$f(z) = f^{-1}(z) = \frac{2}{z}, \qquad g(z) = \frac{1}{2z+1} \qquad \text{and} \qquad g^{-1}(z) = \frac{2z+1}{2z} \ .$$