M436 - Midterm Preview - Enrique Areyan

1.- the line through the points $(1:2:1)$ and $(3:1:0)$ is given by: $\begin{pmatrix}1\\2\\1\end{pmatrix} \times \begin{pmatrix}3\\1\\0\end{pmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 3 & 1 & 0 \end{vmatrix}$

$= \hat{i}[-1] - \hat{j}[-3] + \hat{k}[-5]$

$= \begin{pmatrix}-1\\3\\-5\end{pmatrix}$. Check: $\begin{pmatrix}1\\2\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\3\\-5\end{pmatrix} = -1+6-5 = 0 = \begin{pmatrix}-1\\3\\-5\end{pmatrix} \cdot \begin{pmatrix}3\\1\\0\end{pmatrix} = -3+3$

A line given in homogeneous coordinates as $(a:b:c)$ is incident with the point $(x:y:z)$ iff $(a\ b\ c) \cdot (x\ y\ z) = 0$.

Let us check the options:

(a) $\begin{pmatrix}-2\\-1\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\3\\-5\end{pmatrix} = 2-3-5 \neq 0 \Rightarrow$ not incident.

(b) $\begin{pmatrix}2\\-1\\-1\end{pmatrix} \cdot \begin{pmatrix}-1\\3\\-5\end{pmatrix} = -2-3+5 = 0 \Rightarrow \boxed{\text{incident.}}$

(c) $\begin{pmatrix}2\\1\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\3\\-5\end{pmatrix} = -2+3-5 \neq 0 \Rightarrow$ not incident

2. three lines are concurrent if the determinant of the three homogeneous coefficient are zero. So let us check:

(a) $\det\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 3 \\ 1 & 5 & -3 \end{bmatrix} = 2\det\begin{bmatrix} 1 & 3 \\ 5 & -3 \end{bmatrix} + 3\det\begin{bmatrix} 1 & 3 \\ 1 & -3 \end{bmatrix} + \det\begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$

$= 2\times(-18) + 3\times(-6) + 4 = 2\times(-18) + (-18) + 4$

$= -18[2+1] + 4 \neq 0$

$\Downarrow$ not concurrent

(b) $\det\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 3 \\ 1 & -5 & -3 \end{bmatrix} = 2\det\begin{bmatrix} 1 & 3 \\ -5 & -3 \end{bmatrix} + 3\det\begin{bmatrix} 1 & 3 \\ 1 & -3 \end{bmatrix} + \det\begin{bmatrix} 1 & 1 \\ 1 & -5 \end{bmatrix}$

$= 2\times 12 + 3\times(-6) + -6 = 24 + (-6)[3+1]$

$= 24 - 24 = 0 \Rightarrow \boxed{\text{concurrent}}$

(c) $\det\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 3 \\ 1 & 5 & 3 \end{bmatrix} = 2\det\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} + 3\det\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} + \det\begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$

$= 2\times(-12) + 3\times 0 + 4 = -24 + 4 \neq 0 \Rightarrow$ not concurrent

3. the conic $x^2+4xy-2xz+2yz-z^2=0$ has matrix A given by:

$$A = \begin{pmatrix}1&2&-1\\2&0&1\\-1&1&-1\end{pmatrix}$$ because:

$$(x\ \ y\ \ z)\begin{pmatrix}1&2&-1\\2&0&1\\-1&1&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = (x\ y\ z)\begin{pmatrix}x+2y-z\\2x+z\\-x+y-z\end{pmatrix} =$$

$$= x^2+2xy-xz+2xy+yz-xz+yz-z^2$$

not true $= x^2+4xy-2xz+2yz-z^2$.

A line is tangent to the conic if it satisfies $p^t A p = 0$

(a) $$(0\ \ 1\ \ -1)\begin{pmatrix}1&2&-1\\2&0&1\\-1&1&-1\end{pmatrix}\begin{pmatrix}0\\1\\-1\end{pmatrix} = (0\ \ 1\ \ -1)\begin{pmatrix}3\\-1\\2\end{pmatrix} = -3$$

$\Rightarrow$ not tangent.

(b) $$(1\ \ 1\ \ 0)\begin{pmatrix}1&2&-1\\2&0&1\\-1&1&-1\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix} = (1\ \ 1\ 0)\begin{pmatrix}3\\2\\0\end{pmatrix} = 5$$

$\Rightarrow$ not Tangent

(c) $$(1\ \ -1\ \ 0)\begin{pmatrix}1&2&-1\\2&0&1\\-1&1&-1\end{pmatrix}\begin{pmatrix}1\\-1\\0\end{pmatrix} = (1\ \ -1\ \ 0)\begin{pmatrix}-1\\2\\-2\end{pmatrix} = -3$$

$\Rightarrow$ Not tangent

This method does not seem to work..

Instead, let $p \in A$. We want $Ap =$ given line. Then check $p^t A p \overset{?}{=} 0$.

(a) $$\begin{pmatrix}1&2&-1\\2&0&1\\-1&1&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix} \Leftrightarrow \begin{pmatrix}x+2y-z\\2x+z\\-x+y-z\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$\Rightarrow x+2y-z=0$ ; $2x+z=1$ ; $-x+y-z=-1$ $\Rightarrow z=1-2x$

$\Rightarrow -x+y-1+2x=-1 \Rightarrow x+y=0$

$\Rightarrow y=-x$

$x-2x-1+2x=0 \Rightarrow \boxed{x=1}$

$\Rightarrow \boxed{z=-1} \Rightarrow \boxed{y=-1} \Rightarrow p=(1:-1:-1)$.

$$p^tAp = (1\ -1\ -1)\begin{pmatrix}1&2&-1\\2&0&1\\-1&1&-1\end{pmatrix}\begin{pmatrix}1\\-1\\-1\end{pmatrix} = (1\ -1\ -1)\begin{pmatrix}0\\1\\-1\end{pmatrix} = -1+1 = 0$$

$\Rightarrow$ The line $y-z=0$ is tangent to A through point $(1:-1:-1)$