M447 - Mathematical Models/Applications 1 - Homework 5

Enrique Areyan
October 7, 2014

Chapter 3, Section 3.3

(10) Suppose that a mouse moves in the maze shown in Figure 3.7 and that observations are made every 5 minutes and every time the mouse moves from one compartment to another. Assume that the mouse remains where it is with probability .4 and that whenever it has a choice, it is three times as likely to move to a darker compartment as to a ligther one. In the long run, what is the probability that it is in the compartment with low illumination?

Solution: Since the movement of the mouse depends only on the current location, we can model its behavior with a Markov Chain. We will need four states, each for a compartment the mouse might be in. Let D, L, M, H be label for states Dark, Low, Medium and High.

$$\mathbf{P} = \begin{array}{c} \\ D \\ L \\ M \\ H \end{array} \begin{array}{c} \begin{array}{cccc} D & L & M & H \end{array} \\ \left\| \begin{array}{cccc} 8/20 & 12/20 & 0 & 0 \\ 9/20 & 8/20 & 3/20 & 0 \\ 0 & 9/20 & 8/20 & 3/20 \\ 0 & 0 & 12/20 & 8/20 \end{array} \right\| \end{array}$$

Note that this is an ergodic chain since all states can communicate with each other, and it is aperiodic since from state D we can go back to D in any number of transitions. Hence, we can solve for the limiting distributing as follow: (here states $0, 1, 2, 3$ refer to states D, L, M, H respectively)

$$\pi P = \pi, \text{ and } \sum_{i=0}^{3} \pi_i = 1, \text{ from which we get the equations:}$$

$$\tfrac{8}{20}\pi_0 + \tfrac{9}{20}\pi_1 = \pi_0 \qquad \tfrac{9}{20}\pi_1 = \tfrac{12}{20}\pi_0 \qquad \pi_0 = \frac{9}{12}\pi_1$$

$$\tfrac{12}{10}\pi_0 + \tfrac{8}{20}\pi_1 + \tfrac{9}{20}\pi_2 = \pi_1 \qquad \tfrac{12}{20}\pi_0 + \tfrac{9}{20}\pi_2 = \tfrac{12}{20}\pi_1 \qquad 12\pi_0 + 9\pi_2 = 12\pi_1$$

$$\cancel{\tfrac{3}{20}\pi_1 + \tfrac{8}{20}\pi_2 + \tfrac{12}{20}\pi_3 = \pi_2} \implies \text{(we can ignore this equation)} \implies$$

$$\tfrac{3}{20}\pi_2 + \tfrac{8}{20}\pi_3 = \pi_3 \qquad \tfrac{3}{20}\pi_2 = \tfrac{12}{20}\pi_3 \qquad 3\pi_2 = 12\pi_3$$

We can write each variable in terms of π_1 as follow: replace equation 2 in equation 1: $9\pi_1 + 9\pi_2 = 12\pi_1 \implies \pi_2 = \frac{3}{9}\pi_1$, and replace this equation in equation 3: $\pi_1 = 12\pi_3 \implies \pi_3 = \frac{1}{12}\pi_1$. Now use the normalization equation:

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \implies \frac{9}{12}\pi_1 + \pi_1 + \frac{3}{9}\pi_1 + \frac{1}{12}\pi_1 = 1 \implies \left(\frac{9}{12} + 1 + \frac{3}{9} + \frac{1}{12}\right)\pi_1 = 1 \implies \left(\frac{9 \cdot 22 + 12 \cdot 3}{108}\right)\pi_1 = 1$$

Therefore, $\pi_1 = \dfrac{108}{198 + 36} = \dfrac{108}{234} = \dfrac{6}{13}$. This states corresponds to the state with low illumination, so in the long run the probability that the mouse is in the compartment with low illumination is given by :

$$\boxed{\pi_1 = \frac{6}{13}}$$